Q.1 Verify the addition properties of complex numbers

(i) Closure Property: $\forall z_1, z_2 \in \mathbb{C}$

Solution:

Let $z_1 = a+bi,\ z_2 = c+di$

$$z_1 + z_2 = (a+bi)+(c+di)$$
$$= (a+c)+(b+d)i$$
$$\Rightarrow z_1 + z_2 \in \mathbb{C}$$

Hence, Closure property holds in complex numbers

(ii) Associative Property: $\forall z_1, z_2, z_3 \in \mathbb{C}$

Solution:

Let $z_1 = a+bi, z_2 = c+di, z_3 = e+fi$

$$z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$$

L.H.S $= z_1 + (z_2 + z_3)$

$$= (a+bi)+\left[(c+di)+(e+fi)\right]$$
$$= (a+bi)+\left[(c+e)+(d+f)i\right]$$
$$= \left[a+(c+e)\right]+\left[b+(d+f)\right]i$$
$$= \left[(a+c)+e\right]+\left[(b+d)+f\right]i$$
$$= \left[(a+c)+(b+d)i\right]+(e+fi)$$
$$= \left[(a+bi)+(c+di)\right]+(e+fi)$$
$$= (z_1 + z_2)+z_3$$

= R. H. S

(iii) Identify Element: $\forall z \in \mathbb{C}, \exists O \in \mathbb{C}$

Solution:

Let $z = a+bi$

$$(a+bi)+(0+i0) = (0+i0)+(a+bi) = a+bi$$

(iv) Inverse element: $\forall z \in \mathbb{C}, \exists -z \in \mathbb{C}$

Solution:

Let $z = a+bi \Rightarrow -z = -a-bi$

$$(a+bi)+(-a-bi) = (-a-bi)+(a+bi) = 0$$

(v) Commutative Property: $\forall z_1, z_2 \in \mathbb{C}$

Solution:

Let $z_1 = a+bi,\ z_2 = c+di$

$$z_1 + z_2 = (a+bi)+(c+di)$$
$$= (a+c)+(b+d)i$$
$$= (c+a)+(d+b)i$$
$$= (c+di)+(a+bi)$$
$$= z_2 + z_1$$

Q.2 Verify the multiplication properties of Complex numbers.

(i) Closure Property: $\forall z_1, z_2 \in \mathbb{C}$

Solution:

Let $z_1 = a+bi, z_2 = c+di$

$$z_1.z_2 = (a+bi)(c+di)$$
$$= ac + adi + bci + bdi^2$$
$$= ac + (ad+bc)i - bd$$
$$= (ac-bd)+(ad+bc)i$$
$$\Rightarrow z_1.z_2 \in \mathbb{C}$$

(ii) Associative Property: $\forall z_1, z_2, z_3 \in \mathbb{C}$

Solution:

$$z_1 = a+bi, z_2 = c+di, z_3 = e+fi$$
$$z_1.(z_2.z_3) = (z_1.z_2).z_3$$

L.H.S $= z_1.(z_2.z_3)$

$$= (a+bi).\left[(c+di).(e+fi)\right]$$
$$= (a+bi).\left[ce+cfi+edi+dfi^2\right] \quad \because i^2 = -1$$
$$= (a+bi).\left[ce+(cf+ed)i-df\right]$$
$$= (a+bi).\left[(ce-df)+(cf+ed)i\right]$$
$$= a(ce-df)+a(cf+ed)i+b(ce-df)i+b(cf+ed)i^2$$
$$= a(ce-df)+a(cf+ed)i+b(ce-df)i-b(cf+ed)$$
$$= \left[a(ce-df)-b(cf+ed)\right]+\left[a(cf+ed)+b(ce-df)\right]i$$
$$= (ace-adf-bcf-bde)+(acf+ade+bce-bdf)i \quad \text{(i)}$$

R.H.S $= (z_1.z_2).z_3$

$=[(a+bi)(c+di](e+fi)$

$=\left[ac+adi+bci+bdi^2\right].(e+fi) \quad \because i^2=-1$

$=\left[ac+(ad+bc)i-bd\right].(e+fi)$

$=\left[(ac-bd)+(ad+bc)i\right].(e+fi)$

$=[(ac-bd)e-(ad+bc)f]+[(ac-bd)f+(ad+bc)e]i$

$=(ace-bde-adf-bcf)+(acf-bdf+ade+bce)i$

$=(ace-adf-bcf-bde)+(acf+ade+bce-bdf)i \quad$ (ii)

From equation (i) and equation (ii)

$z_1.(z_2.z_3)=(z_1.z_2).z_3$

(iii) Identity Element: $\forall z\in\mathbb{C}, \exists\, 1+0i\in\mathbb{C}$

Solution:

Let $z=a+bi$

$(a+bi)(1+0i)=(1+0i)(a+bi)=a+bi$

Hence, $1+0i$ is the multiplicative identity of complex number.

(iv) Inverse Element: $\forall z\in\mathbb{C}, \exists \frac{1}{z}\in\mathbb{C}$

with $(z\neq 0)$

Solution:

Let $z=a+bi$

$$\frac{1}{z}=\frac{1}{a+bi}$$

$$=\frac{1}{a+bi}\times\frac{a-bi}{a-bi}$$

$$=\frac{a-bi}{a^2-(bi)^2} \quad \because i^2=-1$$

$$=\frac{a-bi}{a^2-b^2(-1)}$$

$$=\frac{a-bi}{a^2+b^2}$$

$$\frac{1}{z}=\left(\frac{a}{a^2+b^2}\right)+\left(\frac{-b}{a^2+b^2}\right)i$$

Now, we will be show that

$$z.\frac{1}{z}=\frac{1}{z}.z=1$$

$$z.\frac{1}{z}=(a+bi).\left(\frac{a}{a^2+b^2}-\frac{b}{a^2+b^2}\,i\right)$$

$$=\frac{(a)^2-(bi)^2}{a^2+b^2} \quad \because i^2=-1$$

$$=\frac{a^2-b^2(-1)}{a^2+b^2}$$

$$=\frac{a^2+b^2}{a^2+b^2}$$

$$=1$$

Similarly, $\frac{1}{z}.z=1.$

(v) Commutative Property: $\forall z_1,z_2\in\mathbb{C}$

Solution:

Let $z_1=a+bi,\ z_2=c+di$

$z_1.\ z_2=z_2.z_1$

L.H.S $=z_1.z_2$

$=(a+bi)(c+di)$

$=ac+ad\,i+bc\,i+bd\,i^2 \quad \because i^2=-1$

$=(ac-bd)+(ad+bc)i \quad$ (i)

R.H.S $=z_2.z_1$

$=(c+di)(a+bi)$

$=ca+cb\,i+da\,i+db\,i^2 \quad \because i^2=-1$

$=ac+bc\,i+ad\,i+bd\,(-1)$

$=(ac-bd)+(ad+bc)i \quad$ (ii)

By equation (i) and equation (ii) L.H.S = R. H. S

Q.3 Verify the distributive law of complex numbers

$(a,b)\left[(c,d)+(e,f)\right]=(a,b).(c,d)+(a,b)(e,f)$

Proof:

L.H.S $=(a,b)\left[(c,d)+(e,f)\right]$

$=(a,b).(c+e,d+f)$

$\therefore (a,b)(c,d)=(ac-bd,ad+bc)$

$=(a(c+e)-b(d+f),a(d+f)+b(c+e))$

$=(ac+ae+bd-bf,ad+af+bc+be) \quad$ (i)

R.H.S $=(a,b)(c,d)+(a,b)(e,f)$

$=(ac-bd,ad+bc)+(ae-bf,af+be)$

$=(ac-bd+ae-bf,ad+bc+af+be) \quad$ (ii)

By equation (i) and equation (ii) L.H.S = R. H .S

Q.4 Simplify the following.

(i) i^9 ***RWP* 2021**

Solution:

i^9

$=i^8.i$

$=\left(i^2\right)^4.i$

$=\left(-1\right)^4.i$

$=1.i$

$=i$ Answer

(ii) i^{14}

Solution:

i^{14}

$=(i^2)^7$

$=(-1)^7$

$=-1$ Answer

(iii) $(-i)^{19}$

Solution:

$(-i)^{19}$

$=\left(-1\times i\right)^{19}$

$=\left(-1\right)^{19}i^{19}$

$=\left(-1\right)i^{18}.i$

$=\left(-1\right)\left(i^2\right)^9.i$

$=\left(-1\right)\left(-1\right)^9.i$

$=\left(-1\right)\left(-1\right)i$

$=i$ Answer

(iv) $(-1)^{\frac{-21}{2}}$

LHR **2018,** *FSD* **21**

Solution:

$(-1)^{\frac{-21}{2}}$

$=[i^2]^{\frac{-21}{2}}$

$=i^{-21}$

$=\dfrac{1}{i^{21}}$

$=\dfrac{1}{i^{20}.i}$

$=\dfrac{1}{\left(i^2\right)^{10}i}$

$=\dfrac{1}{\left(-1\right)^{10}i}$

$=\dfrac{1}{i}$

$=-i$ Answer

Q.5 Write in terms of i.

(i) $\sqrt{-1}\,b$

Solution:

$=\sqrt{-1}\,b$

$=ib$

(ii) $\sqrt{-5}$

Solution:

$=\sqrt{-5}$

$=\sqrt{-1\times 5}$

$=\sqrt{-1}\ \sqrt{5}$

$=i\sqrt{5}$ Answer

(iii) $\sqrt{\dfrac{-16}{25}}$

Solution:

$=\sqrt{\dfrac{-16}{25}}$

$=\sqrt{\dfrac{-1\times 16}{25}}$

$=\dfrac{\sqrt{-1}\ \sqrt{16}}{\sqrt{25}}$

$=\dfrac{4}{5}i$ Answer

(iv) $\sqrt{\dfrac{1}{-4}}$

Solution:

$=\sqrt{\dfrac{1}{-4}}$

$=\dfrac{1}{\sqrt{-1\times 4}}$

$=\dfrac{1}{\sqrt{-1}\sqrt{4}}$

$=\dfrac{1}{2\,i}$

$=\dfrac{1}{2\,i}\times\dfrac{i}{i}$

$= \dfrac{i}{2i^2}$

$= \dfrac{i}{2(-1)}$

$= \dfrac{-i}{2}$ Answer

Simplify the following:

Q.6 $(7,9)+(3,-5)$

Solution:

$=(7,9)+(3,-5)$

$=(7+3,9+(-5))$

$=(10,4)$ Answer

Q.7 $(8,-5)-(-7,4)$

Solution:

$=(8,-5)-(-7,4)$

$=(8-(-7),-5-4)$

$=(15,-9)$ Answer

Q.8 $(2,6)(3,7)$

Solution:

$=(2,6)(3,7)$

$=(2+6i)(3+7i)$

$=6+14i+18i+42i^2$

$=6+32i+42(-1)$

$=6-42+32i$

$=-36+32i$

$=(-36,32)$ Answer

Q.9 $(5,-4)(-3,-2)$

***LHR* 2022, *MTN* 2023, *GRW* 2023, *SGD* 2021**

Solution:

$=(5,-4)(-3,-2)$

$=(5-4i)(-3-2i)$

$=-15-10i+12i+8i^2$

$=-15+2i+8(-1)$

$=-15+2i-8$

$=-23+2i$

$=(-23,2)$ Answer

Q.10 $(0,3)(0,5)$

Solution:

$=(0,3)(0,5)$

$=(0+3i)(0+5i)$

$=(3i)(5i)$

$=15i^2$

$=15(-1)$

$=-15+0i$

$=(-15,0)$ Answer

Q.11 $(2,6)\div(3,7)$ ***GRW* 2022, *MTN* 2022**

Solution:

$=(2,6)\div(3,7)$

$= \dfrac{2+6i}{3+7i}$

$= \dfrac{2+6i}{3+7i}\times\dfrac{3-7i}{3-7i}$

$= \dfrac{(2+6i)(3-7i)}{(3+7i)(3-7i)}$

$= \dfrac{6-14i+18i-42i^2}{3^2-(7i)^2}$

$= \dfrac{6+4i-42(-1)}{9-49(-1)}$

$= \dfrac{6+4i+42}{9+49}$

$= \dfrac{48+4i}{58}$

$= \dfrac{48}{58}+\dfrac{4}{58}i$

$= \dfrac{24}{29}+\dfrac{2}{29}i$

$= \left(\dfrac{24}{29},\dfrac{2}{29}\right)$ Answer

Q.12 $(5,-4)\div(-3,-8)$ ***LHR* 2021**

Solution:

$=(5,-4)\div(-3,-8)$

$= \dfrac{5-4i}{-3-8i}$

$$= \frac{5-4i}{-3-8i} \times \frac{-3+8i}{-3+8i}$$

$$= \frac{-15+40i+12i-32i^2}{(-3)^2-(8i)^2}$$

$$= \frac{-15+52i-32(-1)}{9-64(-1)}$$

$$= \frac{-15+52i+32}{9+64}$$

$$= \frac{17+52i}{73}$$

$$= \frac{17}{73} + \frac{52}{73}i$$

$$= \left(\frac{17}{73}, \frac{52}{73}\right)$$ Answer

Q.13 Prove that the sum as well as the product of any two conjugate complex numbers is a real number. ***FSD* 2022, *GRW* 2019**

Proof:

Let $z = x + iy$ and $\bar{z} = x - iy$

$$z + \bar{z} = x + i\cancel{y} + x - i\cancel{y}$$

$= 2x$ is a real number

$$z.\bar{z} = (x+iy)(x-iy)$$

$= (x)^2 - (iy)^2 \quad \because a^2 - b^2 = (a+b)(a-b)$

$= x^2 - i^2 y^2$

$= x^2 - (-1) y^2$

$= x^2 + y^2$ is a real number.

Q.14 Find the multiplicative inverse of each of the following numbers:

(i) $(-4,7)$

***LHR* 2018, *SGD* 2021-22, *RWP* 2023**

Solution:

Let $z = (-4,7)$

$$z = -4 + 7i$$

$$\frac{1}{z} = \frac{1}{-4+7i}$$

$$= \frac{1}{-4+7i} \times \frac{-4-7i}{-4-7i}$$

$$= \frac{-4-7i}{(-4)^2-(7i)^2}$$

$$= \frac{-4-7i}{16-49(-1)}$$

$$= \frac{-4-7i}{16+49}$$

$$= \frac{-4-7i}{65}$$

$$= \frac{-4}{65} - \frac{7}{65}i$$

$$\frac{1}{z} = \left(\frac{-4}{65}, \frac{-7}{65}\right)$$ Answer

(ii) $(\sqrt{2}, -\sqrt{5})$

***GRW* 2021-22, *LHR* 2019-22, *FSD* 2021**

Solution:

Let $z = (\sqrt{2}, -\sqrt{5})$

$$z = \sqrt{2} - \sqrt{5}i$$

$$\frac{1}{z} = \frac{1}{\sqrt{2} - \sqrt{5}i}$$

$$= \frac{1}{\sqrt{2}-\sqrt{5}i} \times \frac{\sqrt{2}+\sqrt{5}i}{\sqrt{2}+\sqrt{5}i}$$

$$= \frac{\sqrt{2}+\sqrt{5}i}{(\sqrt{2})^2-(\sqrt{5}i)^2}$$

$$\frac{1}{z} = \frac{\sqrt{2}+\sqrt{5}i}{2-5(-1)}$$

$$= \frac{\sqrt{2}+\sqrt{5}i}{2+5}$$

$$= \frac{\sqrt{2}+\sqrt{5}i}{7}$$

$$= \frac{\sqrt{2}}{7} + \frac{\sqrt{5}}{7}i$$

$$\frac{1}{z} = \left(\frac{\sqrt{2}}{7}, \frac{\sqrt{5}}{7}\right)$$ Answer

(iii) $(1,0)$

Solution:

Let $z = (1, 0)$

$$z = 1 + 0i$$

$$\frac{1}{z} = \frac{1}{1+0i}$$

$$\frac{1}{z} = \frac{1}{1+0i} \times \frac{1-0i}{1-0i}$$
$$= \frac{1-0i}{(1)^2-(0i)^2}$$
$$= \frac{1}{1}$$
$$\frac{1}{z} = 1$$
$$\frac{1}{z} = (1,0) \text{ Answer}$$

Q.15 Factorize the following

(i) a^2+4b^2

***LHR* 2019, *GRW* 2021, *SGD* 2018**

Solution:

$$= a^2+4b^2$$
$$= a^2-\left(-4b^2\right)$$
$$= a^2-\left(-1\right)4b^2$$
$$= a^2-i^2 4b^2$$
$$= a^2-(i2b)^2$$
$$= (a-i2b)(a+i2b) \text{ Answer}$$

(ii) $9a^2+16b^2$

***FSD* 2018-22, *LHR* 2022, *DGK* 2022, *GRW* 2023**

Solution:

$$9a^2+16b^2$$
$$= 9a^2-(-16b^2)$$
$$= 9a^2-(-1)16b^2$$
$$= 9a^2-i^2 16b^2$$
$$= (3a)^2-(i\ 4b)^2$$
$$= \left(3a-i4b\right)\left(3a+i4b\right) \text{ Answer}$$

(iii) $3x^2+3y^2$ ***SHW* 2023**

Solution:

$$= 3x^2+3y^2$$
$$= 3[x^2+y^2]$$
$$= 3[x^2-(-y^2)]$$
$$= 3[x^2-(-1)y^2]$$
$$= 3[x^2-i^2y^2]$$
$$= 3[x^2-(iy)^2]$$
$$= 3\left(x-iy\right)\left(x+iy\right) \text{ Answer}$$

Q.16 Separate into real and imaginary parts (Write as a simple complex number):

(i) $\dfrac{2-7i}{4+5i}$ ***LHR* 2021, *SGD* 2022**

Solution:

$$= \frac{2-7i}{4+5i}$$
$$= \frac{2-7i}{4+5i} \times \frac{4-5i}{4-5i}$$
$$= \frac{(2-7i)(4-5i)}{(4)^2-(5i)^2}$$
$$= \frac{8-10i-28i+35i^2}{16-25(-1)}$$
$$= \frac{8-38i+35(-1)}{16+25}$$
$$= \frac{8-38i-35}{41}$$
$$= \frac{-27-38i}{41}$$
$$= \frac{-27}{41} - \frac{38}{41}i$$

Real Part $= \dfrac{-27}{41}$, Imaginary Part $= \dfrac{-38}{41}$

(ii) $\dfrac{(-2+3i)^2}{1+i}$ ***DGK* 2022**

Solution:

$$= \frac{(-2+3i)^2}{1+i}$$
$$= \frac{(-2)^2+(3i)^2+2(-2)(3i)}{1+i}$$
$$= \frac{4-9-12i}{1+i}$$
$$= \frac{-5-12i}{1+i}$$
$$= \frac{-5-12i}{1+i} \times \frac{1-i}{1-i}$$
$$= \frac{(-5-12i)(1-i)}{(1+i)(1-i)}$$
$$= \frac{-5+5i-12i+12i^2}{1^2-i^2}$$

$$= \frac{-5-7i+12(-1)}{1-(-1)}$$

$$= \frac{-5-7i-12}{1+1}$$

$$= \frac{-17-7i}{2}$$

$$= \frac{-17}{2} - \frac{7}{2}i$$

Real Part $= \frac{-17}{2}$, Imaginary Part $= \frac{-7}{2}$

(iii) $\frac{i}{1+i}$

***GRW* 2019, *RWP* 2018-23, *MTN* 2022-23, *SHW* 2022)**

<u>Solution</u>:

$$= \frac{i}{1+i}$$

$$= \frac{i}{1+i} \times \frac{1-i}{1-i}$$

$$= \frac{i(1-i)}{1^2-i^2}$$

$$= \frac{i-i^2}{1-(-1)}$$

$$= \frac{i-(-1)}{2}$$

$$= \frac{1+i}{2}$$

$$= \frac{1}{2} + \frac{1}{2}i$$

Real Part $= \frac{1}{2}$, Imaginary Part $= \frac{1}{2}$