# Mathematics-11

## Exercise - 2.8



**Q.1 Operation ⊕ is performed on the two-member set G={0,1} is shown in the adjoining table. Answer the questions.**

**(i) Name the identity element if it exists.**

| ⊕ | 0 | 1 |
|---|---|---|
| 0 | 0 | 1 |
| 1 | 1 | 0 |

**Solution:** From the given table, $0+0=0,\ 0+1=1$ which shows that 0 is the identity element.

**(ii) What is the inverse of 1?**

**Solution:** Since $1+1=0$ (identity) so inverse of 1 is 1.

**(iii) Is the set G, under the given operation a group? Abelian or non-Abelian?**

**Solution:** The numbers in table satisfy all the properties of abelian group so $G$ is an abelian group under addition.

**Q.2 The operation ⊕ as performed on the set $\{0,1,2,3\}$ is shown in the adjoining table, show that the set is an abelian group.**

**Solution:**

| ⊕ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 |
| 1 | 1 | 2 | 3 | 0 |
| 2 | 2 | 3 | 0 | 1 |
| 3 | 3 | 0 | 1 | 2 |

**(i)** Each element of the table is an element of the given set $\{0,1,2,3\}$, so closure law holds.

**(ii)** Associative law holds.

e.g. (1 + 2) + 3 = 1 + (2 + 3)

3 + 3 = 1 + 1

2 = 2

**(iii)** 0 is the identity element.

**(iv)** The inverse of 0 is 0.

The inverse of 1 is 3.

The inverse of 2 is 2.

The inverse of 3 is 1.

so inverse of every element exists.

**(v)** Commutative law also holds. e.g. 1 + 2 = 2 + 1

3 = 3

So the set $\{0,1,2,3\}$ is an abelian group under addition modulo 4.

**Q.3 For each of the following sets, determine whether or not the set forms a group with respect to the indicated operation.**

| **Set** | **Operations** |
|---|---|
| **(i)** The set of rational numbers | $\times$ |
| **(ii)** The set of rational numbers | $+$ |
| **(iii)** The set of positive rational numbers | $\times$ |
| **(iv)** The set of integers | $+$ |
| **(v)** The set of integers | $\times$ |

**Solution:**

**(i)** Let $Q =$ The set of rational numbers

(i) As product of any two rational numbers is also a rational number so $Q$ is closed w.r.t. multiplication

(ii) Multiplication of rational numbers is always associative

i.e. $\forall a,b,c \in Q \Rightarrow (ab)c = a(bc)$

(iii) Here identity element is $1 \in Q$

(iv) Multiplicative inverse of $0 \in Q$ does not exist, so $Q$ is not a group under multiplication

**(ii)** Let $Q =$ The set of rational numbers

(i) As sum of any two rational numbers is also a rational number so $Q$ is closed w.r.t. addition.

(ii) Addition of rational numbers is always associative.

i.e. $\forall a,b,c \in Q \Rightarrow (a+b)+c = a+(b+c)$

(iii) Here identity element is $0 \in Q$

(iv) $\forall a \in Q$, the additive inverse is $-a \in Q$.

Hence $Q$ is a group under addition

**(iii)** Let $Q^+ =$ The set of positive rational numbers

(i) As product of any two positive rational numbers is also a positive rational number so $Q^+$ is closed w.r.t. multiplication.

(ii) Multiplication of positive rational numbers is always associative.

i.e. $\forall a,b,c \in Q^+ \Rightarrow (ab)c = a(bc)$

(iii) Here identity element is $1 \in Q^+$

(iv) $\forall a \in Q^+$, the multiplicative inverse is $\frac{1}{a} \in Q^+$

Hence $Q^+$ is a group under multiplication

**(iv)** Let $Z =$ The set of integers

**(i)** As sum of any two integers is also an integer so $Z$ is closed w.r.t. addition.

**(ii)** Addition of integers is always associative

i.e. $\forall a,b,c \in Z \Rightarrow (a+b)+c = a+(b+c)$

**(iii)** Here identity element is $0 \in Z$

**(iv)** $\forall a \in Z$, the additive inverse is $-a \in Z$.

Hence $Z$ is a group under addition

**(v)** Let $Z =$ The set of integers

**(i)** As product of any two integers is also an integer so $Z$ is closed w.r.t. multiplication

**(ii)** Multiplication of integers is always associative

i.e. $\forall a,b,c \in Z \Rightarrow (ab)c = a(bc)$

**(iii)** Here identity element is $1 \in Z$

**(iv)** Multiplicative inverse of any element of Z does not exist in Z except $\pm 1 \in Z$.

Hence $Z$ is not a group under multiplication.

**Q.4 Show that the adjoining table represents the sums of the elements of the set {E,O}.What is the identity element of this set? Show that this set is an abelian group.**

| $\oplus$ | E | O |
|---|---|---|
| E | E | O |
| O | O | E |

**Solution:** Since the sum of two even integers is also an even integer, so $E+E=E$

The sum of an even and an odd integers is odd, i.e. $E+O=O$

The sum of two odd integers is also even, i.e. $O+O=E$

Hence given table represents the sums of the elements of the set $\{E,O\}$.

Now since $E+E=E$ and $E+O=O$ so E is the identity element.

Now we show that this is an abelian group.

**(i)** Given set is closed under addition.

**(ii)** Associative law of addition holds in given set. e.g. $(E+O)+E = E+(O+E)$

$O+E = E+O$

$O=O$

**(iii)** Already proved that identity is E, so identity exists.

**(iv)** The inverse of E is E.

The inverse of O is O.

so inverse of each element exists.

**(v)** $O+E=O=E+O$ so commutative law holds.

Hence given set is an abelian group under addition.

**Q.5 Show that the set $\{1,\omega,\omega^2\}$, when $\omega^3=1$, is an Abelian group w. r. t. ordinary multiplication.**

**Solution:** Let $G=\{1,\omega,\omega^2\}$

| $\times$ | $1$ | $\omega$ | $\omega^2$ |
|---|---|---|---|
| $1$ | $1$ | $\omega$ | $\omega^2$ |
| $\omega$ | $\omega$ | $\omega^2$ | $1$ |
| $\omega^2$ | $\omega^2$ | $1$ | $\omega$ |

**(i)** From the table it is clear that $G$ is closed w.r.t. $\times$.

**(ii)** Multiplication of complex numbers is associative and $G\subset C$, so associative law of multiplication holds in $G$.

**(iii)** Identity element of $G$ is 1

**(iv)** Inverse of each element exists.

Inverse of 1 is 1

Inverse of $\omega$ is $\omega^2$

Inverse of $\omega^2$ is $\omega$

**(v)** Multiplication of complex numbers is commutative and $G\subset C$, so commutative law of multiplication holds in $G$

Hence $G$ is an abelian group w.r.t. multiplication.

**Q.6 If G is a group under operation $*$ and $a,b\in G$, find the solutions of the equations**

**(i)** $a*x=b$

**(ii)** $x*a=b$

**Solution:**

**(i)** Since $a\in G$ and G is a group so $a^{-1}\in G$

Given $a*x=b$

$\Rightarrow a^{-1}*(a*x)=a^{-1}*b$

$\Rightarrow (a^{-1}*a)*x=a^{-1}*b$ (Associative Law)

$\Rightarrow e*x=a^{-1}*b$ $(a^{-1}*a=e)$

$\Rightarrow x = a^{-1} * b$ $\quad (e * x = x)$

**(ii)** Since $a \in G$ and G is a group so $a^{-1} \in G$

Given $x * a = b$

$\Rightarrow (x * a) * a^{-1} = b * a^{-1}$

$\Rightarrow x * (a * a^{-1}) = b * a^{-1}$ (Associative Law)

$\Rightarrow x * e = b * a^{-1}$ $\quad (a * a^{-1} = e)$

$\Rightarrow x = b * a^{-1}$ $\quad (x * e = x)$

**Q.7 Show that the set consisting of elements of the form $a + \sqrt{3}b$, ($a,b$ being rational) is an abelian group w .r .t addition.**

**Solution:** Let $G = \{a + \sqrt{3}b \mid a, b \in Q\}$

Let $x, y, z$ be any three elements of $G$ and

$x = a + \sqrt{3}b$, $\quad y = c + \sqrt{3}d$, $\quad z = e + \sqrt{3}f$ where $a, b, c, d, e, f$ are rational numbers.

**(i)** $x + y = (a + \sqrt{3}b) + (c + \sqrt{3}d)$

$= (a + c) + \sqrt{3}(b + d) \in G$ as $a + c, b + d \in Q$

So G is closed under addition.

**(ii)** Addition of real numbers is associative and $G \subset R$ so associative law of addition holds in $G$

**(iii)** $0 = 0 + \sqrt{3}(0)$ is the identity element in $G$.

**(iv)** For all $x = a + \sqrt{3}b \in G$, we have $-x = -a - \sqrt{3}b \in G$ such that

$x + (-x) = (a + \sqrt{3}b) + (-a - \sqrt{3}b) = 0$. This shows that inverse of each element of $G$ exists in $G$.

**(v)** Addition of real number is commutative and $G \subset R$ so commutative law of addition holds in $G$.

Hence $G$ is an abelian group under addition

**Q.8 Determine whether $(P(S), *)$ where $*$ stands for intersection is a semi-group, a monoid or neither. If it is a monoid, specify its identity.**

**(i)** Since the intersection of two subsets of *S* is also its subset and will be contained by P(S), so *P*(*S*) is closed.

**(ii)** Intersection of sets is always associative.

i.e. $\forall A, B, C \in P(S) \Rightarrow (A \cap B) \cap C = A \cap (B \cap C)$

**(iii)** For all $A \in P(S)$, $A \cap S = A$ (Q*A* is a subset of *S*). This shows that the identity element is $S \in P(S)$.

This shows that $(P(S), *)$ is a monoid having identity *S*.

**Q.9 Complete the following table to obtain a semi-group under $*$.**

| $*$ | $a$ | $b$ | $c$ |
|---|---|---|---|
| $a$ | $c$ | $a$ | $b$ |
| $b$ | $a$ | $b$ | $c$ |
| $c$ | | | $a$ |

**Solution:**

Let missing elements be $p$ and $q$

| $*$ | $a$ | $b$ | $c$ |
|---|---|---|---|
| $a$ | $c$ | $a$ | $b$ |
| $b$ | $a$ | $b$ | $c$ |
| $c$ | $p$ | $q$ | $a$ |

$$\begin{aligned} p &= c*a \\ &= (a*a)*a \quad (c = a*a) \\ &= a*(a*a) \quad (\text{Associative Law}) \\ &= a*c \\ &= b \end{aligned}$$

$$\begin{aligned} q &= c*b \\ &= (a*a)*b \quad (c = a*a) \\ &= a*(a*b) \quad (\text{Associative Law}) \\ &= a*a \\ &= c \end{aligned}$$

**Q.10 Prove that all 2×2 non-singular matrices over the real field form a non-abelian group under multiplication.**

**Solution:** Let G be the set of all 2×2 non-singular matrices over the real field.

**(i)** As product of any two 2 × 2 matrices is again a matrix of order 2 × 2, so G is closed under multiplication.

**(ii)** Associative law of multiplication holds in matrices confirmable for multiplication.
i.e. $\forall\ A, B, C \in G \Rightarrow (AB)C = A(BC)$.

**(iii)** Since identity matrix of order 2×2 is also a non – singular matrix, so $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in G$ is identity element in $G$.

**(iv)** The inverse of every 2×2 non-singular matrix exists and is given by $A^{-1} = \dfrac{AdjA}{|A|} \in G$, so inverse of every matrix of G exists.

**(v)** Commutative law of multiplication does not hold in matrices i.e. generally, $AB \neq BA$.
So G is a non-abelian group under multiplication.