

EXERCISE 2.2

Q.1 Find from first principles, the derivatives of the following expressions w.r.t. their respective independent variables.

(i) $(ax+b)^3$

Solution:

Let $y = (ax+b)^3$

Taking increments on both sides,

$$y + \delta y = [a(x + \delta x) + b]^3$$

$$\delta y = [a(x + \delta x) + b]^3 - y$$

$$\delta y = (ax + b + a\delta x)^3 - (ax + b)^3$$

$$\delta y = \left[\cancel{(ax+b)^3} + 3(ax+b)^2 \cdot a\delta x + 3(ax+b)(a\delta x)^2 + (a\delta x)^3 \right] - \cancel{(ax+b)^3}$$

$$\delta y = 3a(ax+b)^2 \delta x + 3a^2(ax+b)\delta x^2 + a^3\delta x^3$$

Dividing both sides by ' δx '

$$\frac{\delta y}{\delta x} = 3a(ax+b)^2 + 3a^2(ax+b)\delta x + a^3\delta x^2$$

Taking limit as $\delta x \rightarrow 0$

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} [3a(ax+b)^2 + 3a^2(ax+b)\delta x + a^3\delta x^2]$$

$$\boxed{\frac{dy}{dx} = 3a(ax+b)^2}$$

(ii) $(2x+3)^5$

Solution:

Let $y = (2x+3)^5$

Taking increments on both sides,

$$y + \delta y = [2(x + \delta x) + 3]^5$$

$$\delta y = (2x + 2\delta x + 3)^5 - (2x + 3)^5$$

$$\delta y = [(2x+3) + 2\delta x]^5 - (2x+3)^5$$

$$\delta y = \binom{5}{0}(2x+3)^5 + \binom{5}{1}(2x+3)^4 \cdot (2\delta x) + \binom{5}{2}(2x+3)^3 (2\delta x)^2$$

$$+ \binom{5}{3}(2x+3)^2 (2\delta x)^3 + \binom{5}{4}(2x+3)(2\delta x)^4 + \binom{5}{5}(2\delta x)^5 - (2x+3)^5$$

Dividing both sides by ' δx '

$$\frac{\delta y}{\delta x} = \cancel{\binom{5}{0}(2x+3)^5} + \binom{5}{1}(2x+3)^4 \cdot 2 + \binom{5}{2}(2x+3)^3 \cdot 4\delta x$$

Unit- 2

Differentiation

$$+ \binom{5}{3} (2x+3)^2 \cdot 8\delta x^2 + \binom{5}{4} (2x+3) 16\delta x^3 + \binom{5}{5} \delta x^4 - \cancel{(2x+3)^5}$$

Taking limit as $\delta x \rightarrow 0$

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = (2)(5)(2x+3)^4$$

$$\boxed{\frac{dy}{dx} = 10(2x+3)^4}$$

(iii) $(3t+2)^{-2}$

Solution:

Let $y = (3t+2)^{-2}$

Taking increments on both sides,

$$y + \delta y = [3(t + \delta t) + 2]^{-2}$$

$$\delta y = (3t + 3\delta t + 2)^{-2} - (3t + 2)^{-2}$$

$$\delta y = (3t + 2 + 3\delta t)^{-2} - (3t + 2)^{-2}$$

$$\delta y = (3t + 2)^{-2} \left(1 + \frac{3\delta t}{3t+2} \right)^{-2} - (3t + 2)^{-2}$$

$$\delta y = (3t + 2)^{-2} \left[\left(1 + \frac{3\delta t}{3t+2} \right)^{-2} - 1 \right]$$

$$\delta y = (3t + 2)^{-2} \left[1 + (-2) \left(\frac{3\delta t}{3t+2} \right) + \frac{(-2)(-3)}{2} \left(\frac{3\delta t}{3t+2} \right)^2 + \dots \right]$$

Dividing both sides by ' δt '

$$\frac{\delta y}{\delta t} = (3t + 2)^{-2} \left[\frac{-6}{3t+2} + \frac{27\delta t}{(3t+2)^2} + \dots \right]$$

Taking limit as $\delta t \rightarrow 0$

$$\lim_{\delta t \rightarrow 0} \frac{\delta y}{\delta t} = \lim_{\delta t \rightarrow 0} (3t + 2)^{-2} \left[\frac{-6}{3t+2} + \frac{27\delta t}{(3t+2)^2} + \dots \right]$$

$$\boxed{\frac{dy}{dt} = \frac{-6}{(3t+2)^3}}$$

(iv) $(ax+b)^{-5}$

Solution:

Let $y = (ax+b)^{-5}$

Taking increments on both sides,

$$y + \delta y = [a(x + \delta x) + b]^{-5}$$

Unit- 2

Differentiation

$$y + \delta y = [a(x + \delta x) + b]^{-5}$$

$$\delta y = (ax + a\delta x + b)^{-5} - (ax + b)^{-5}$$

$$\delta y = (ax + b + a\delta x)^{-5} - (ax + b)^{-5}$$

$$\delta y = (ax + b)^{-5} \left(1 + \frac{a\delta x}{ax + b} \right)^{-5} - (ax + b)^{-5}$$

$$\delta y = (ax + b)^{-5} \left[\left(1 + \frac{a\delta x}{ax + b} \right)^{-5} - 1 \right]$$

$$\delta y = (ax + b)^{-5} \left[1 + (-5) \left(\frac{a\delta x}{ax + b} \right) + \frac{(-5)(-6)}{2} \left(\frac{a\delta x}{ax + b} \right)^2 + \dots \right]$$

Dividing both sides by ' δx '

$$\frac{\delta y}{\delta x} = (ax + b)^{-5} \left[\frac{-5a}{ax + b} + \frac{15a^2\delta x}{(ax + b)^2} + \dots \right]$$

Taking limit as $\delta x \rightarrow 0$

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} (ax + b)^{-5} \left[\frac{-5a}{ax + b} + \frac{15a^2\delta x}{(ax + b)^2} + \dots \right]$$

$$\boxed{\frac{dy}{dx} = \frac{-5a}{(ax + b)^6}}$$

(v) $\frac{1}{(az - b)^7}$

Solution:

$$\text{Let } y = (az - b)^{-7}$$

Taking increments on both sides,

$$y + \delta y = [a(z + \delta z) - b]^{-7}$$

$$\delta y = (a(z + \delta z) - b)^{-7} - y$$

$$\delta y = (az + a\delta z - b)^{-7} - (az - b)^{-7}$$

$$\delta y = (az - b + a\delta z)^{-7} - (az - b)^{-7}$$

$$\delta y = (az - b)^{-7} \left[1 + \frac{a\delta z}{az - b} \right]^{-7} - (az - b)^{-7}$$

$$\delta y = (az - b)^{-7} \left[\left(1 + \frac{a\delta z}{az - b} \right)^{-7} - 1 \right]$$

Unit- 2

Differentiation

$$\delta y = (az - b)^{-7} \left[1 + \frac{(-7)a\delta z}{az - b} + \frac{(-7)(-8)}{2} \frac{a^2 \delta z^2}{(az - b)^2} + \dots \right]$$

Dividing both sides by ' δz '

$$\frac{\delta y}{\delta z} = (az - b)^{-7} \left[1 + \frac{-7a}{az - b} + \frac{28a^2 \delta z}{(az - b)^2} + \dots \right]$$

Taking limit as $\delta z \rightarrow 0$

$$\lim_{\delta z \rightarrow 0} \frac{\delta y}{\delta z} = \lim_{\delta z \rightarrow 0} (az - b)^{-7} \left[\frac{-7a}{(az - b)} + \frac{28a^2 \delta z}{(az - b)^2} + \dots \right]$$

$$\boxed{\frac{dy}{dz} = \frac{-7a}{(az - b)^8}}$$

Theorems on differentiation:

$$(1) \frac{d}{dx}(c) = 0 \quad (c \text{ is constant})$$

$$(2) \frac{d}{dx}(x^n) = nx^{n-1}, \quad n \in Q$$

$$(3) \frac{d}{dx}[cf(x)] = c \cdot f'(x)$$

Proof:

$$(3) \text{ Let } y = c \cdot f(x)$$

$$y + \delta y = cf(x + \delta x)$$

$$\delta y = cf(x + \delta x) - cf(x)$$

Dividing both sides by ' δx '

$$\frac{\delta y}{\delta x} = c \left[\frac{f(x + \delta x) - f(x)}{\delta x} \right]$$

Taking limit as $\delta x \rightarrow 0$

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = c \left[\lim_{\delta x \rightarrow 0} \frac{f(x + \delta x) - f(x)}{\delta x} \right]$$

$$\frac{d}{dx}(y) = cf'(x)$$

$$\frac{d}{dx}(cf(x)) = c \cdot f'(x)$$

$$(4) \frac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x)$$

Let

$$y = f(x) + g(x)$$

$$y + \delta y = f(x + \delta x) + g(x + \delta x)$$

$$\delta y = f(x + \delta x) + g(x + \delta x) - [f(x) + g(x)]$$

Unit- 2

Differentiation

$$= f(x + \delta x) - f(x) + g(x + \delta x) - g(x)$$

Dividing both sides by ' δx '

$$\frac{\delta y}{\delta x} = \frac{f(x + \delta x) - f(x)}{\delta x} + \frac{g(x + \delta x) - g(x)}{\delta x}$$

Taking limit as $\delta x \rightarrow 0$

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} \frac{f(x + \delta x) - f(x)}{\delta x} + \lim_{\delta x \rightarrow 0} \frac{g(x + \delta x) - g(x)}{\delta x}$$

$$\frac{dy}{dx} = f'(x) + g'(x)$$

$$\frac{d}{dx} [f(x) + g(x)] = f'(x) + g'(x)$$

The proof for $\frac{d}{dx} [f(x) - g(x)] = f'(x) - g'(x)$ is similar.

(5) The Product Rule:

$$\frac{d}{dx} [f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$$

Proof:

$$\text{Let } y = f(x)g(x)$$

$$y + \delta y = f(x + \delta x)g(x + \delta x)$$

$$= f(x + \delta x)g(x + \delta x) - y$$

$$\delta y = f(x + \delta x)g(x + \delta x) - f(x)g(x)$$

Subtracting and adding $f(x)g(x + \delta x)$

$$\delta y = f(x + \delta x)g(x + \delta x) - f(x)g(x + \delta x) + f(x)g(x + \delta x) - f(x)g(x)$$

$$\delta y = [f(x + \delta x) - f(x)]g(x + \delta x) + f(x)[g(x + \delta x) - g(x)]$$

Dividing both sides by ' δx '

$$\frac{\delta y}{\delta x} = \left[\frac{f(x + \delta x) - f(x)}{\delta x} \right] g(x + \delta x) + f(x) \left[\frac{g(x + \delta x) - g(x)}{\delta x} \right]$$

Taking limit as $\delta x \rightarrow 0$

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \left[\lim_{\delta x \rightarrow 0} \frac{f(x + \delta x) - f(x)}{\delta x} \right] \lim_{\delta x \rightarrow 0} g(x + \delta x) + f(x) \lim_{\delta x \rightarrow 0} \left[\frac{g(x + \delta x) - g(x)}{\delta x} \right]$$

$$\frac{dy}{dx} = f'(x)g(x) + f(x)g'(x)$$

(6) The Quotient Rule:

Unit- 2

Differentiation

$$\frac{d}{dx} \left(\frac{f(x)}{g(x)} \right) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}, g(x) \neq 0$$

Proof:

$$\text{Let } y = \frac{f(x)}{g(x)}$$

$$y + \delta y = \frac{f(x + \delta x)}{g(x + \delta x)}$$

$$\delta y = \frac{f(x + \delta x)}{g(x + \delta x)} - y$$

$$\begin{aligned} \delta y &= \frac{f(x + \delta x)}{g(x + \delta x)} - \frac{f(x)}{g(x)} \\ &= \frac{f(x + \delta x)g(x) - f(x)g(x + \delta x)}{g(x + \delta x)g(x)} \end{aligned}$$

Subtracting and adding $f(x)g(x)$

$$\begin{aligned} &= \frac{f(x + \delta x)g(x) - f(x)g(x) - f(x)g(x + \delta x) + f(x)g(x)}{g(x + \delta x)g(x)} \\ &= \frac{1}{g(x + \delta x)g(x)} \left[\{f(x + \delta x)g(x) - f(x)g(x)\} - \{f(x)g(x + \delta x) - f(x)g(x)\} \right] \end{aligned}$$

Dividing both sides by ' δx '

$$\frac{\delta y}{\delta x} = \frac{1}{g(x + \delta x)g(x)} \left[g(x) \frac{f(x + \delta x) - f(x)}{\delta x} - f(x) \frac{g(x + \delta x) - g(x)}{\delta x} \right]$$

Taking limit as $\delta x \rightarrow 0$

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} \frac{1}{g(x + \delta x)g(x)} \left[g(x) \lim_{\delta x \rightarrow 0} \frac{f(x + \delta x) - f(x)}{\delta x} - f(x) \lim_{\delta x \rightarrow 0} \frac{g(x + \delta x) - g(x)}{\delta x} \right]$$

$$\frac{dy}{dx} = \frac{1}{g(x)g(x)} [g(x)f'(x) - f(x)g'(x)]$$

$$= \frac{g(x)f'(x) - f(x)g'(x)}{[g^2(x)]^2}$$

(7) The reciprocal rule:

If g is differentiable at x and $g(x) \neq 0$ then $\frac{1}{g(x)}$ is differentiable and

$$\frac{d}{dx} \left[\frac{1}{g(x)} \right] = -\frac{g'(x)}{[g(x)]^2}$$