EXERCISE 2.4

Q.1 Find $\frac{dy}{dx}$ by making suitable substitutions in the following functions defined as:

(i) $y=\sqrt{\frac{1-x}{1+x}}$

Solution:

Let $t=\frac{1-x}{1+x}$

then $y=\sqrt{t}$

$$\frac{dy}{dt}=\frac{1}{2\sqrt{t}}\ldots(i)$$

$$\frac{dy}{dt}=\frac{1}{2}\sqrt{\frac{1+x}{1-x}}$$

Now $\frac{dt}{dx}=\frac{(1+x)(-1)-(1-x)(1)}{(1+x)^2}$

$$\frac{dt}{dx}=\frac{-1-\cancel{x}-1+\cancel{x}}{(1+x)^2}$$

$$\frac{dt}{dx}=\frac{-2}{(1+x)^2}\ \ldots(ii)$$

Applying chain rule on (i) & (ii)

$$\frac{dy}{dx}=\frac{dy}{dt}.\frac{dt}{dx}$$

$$\frac{dy}{dx}=\frac{1}{\cancel{2}}\sqrt{\frac{1+x}{1-x}}.\frac{-\cancel{2}}{(1+x)^2}$$

$$\boxed{\frac{dy}{dx}=-\frac{1}{\sqrt{1-x}.(1+x)^{\frac{3}{2}}}}$$

(ii) $y=\sqrt{x+\sqrt{x}}$

Solution:

Let $t=x+\sqrt{x}$

then $y=\sqrt{t}$

$$\frac{dy}{dt}=\frac{1}{2\sqrt{t}}$$

$$\frac{dy}{dt}=\frac{1}{2}\frac{1}{\sqrt{x+\sqrt{x}}}\ \ldots(i)$$

also, $t=x+\sqrt{x}$

$$\frac{dt}{dx}=1+\frac{1}{2\sqrt{x}}\ \ldots(ii)$$

Applying chain rule on equation (i) and equation (ii),

$$\frac{dy}{dx}=\frac{dy}{dt}.\frac{dt}{dx}$$

$$\frac{dy}{dx}=\frac{1}{2\sqrt{x+\sqrt{x}}}.\frac{1+2\sqrt{x}}{2\sqrt{x}}$$

$$\boxed{\frac{dy}{dx}=\frac{1+2\sqrt{x}}{4\sqrt{x}\sqrt{x+\sqrt{x}}}}$$

(iii) $y=x\sqrt{\frac{a+x}{a-x}}$

Solution:

Let $t=\frac{a+x}{a-x}$ then $y=xt^{\frac{1}{2}}$

$$t=\frac{a+x}{a-x}$$

$$\frac{dt}{dx}=\frac{(a-x)(1)-(a+x)(-1)}{(a-x)^2}$$

$$\frac{dt}{dx}=\frac{a-\cancel{x}+a+\cancel{x}}{(a-x)^2}$$

$$\frac{dt}{dx}=\frac{2a}{(a-x)^2}\ \ldots(i)$$

Now, $y=xt^{\frac{1}{2}}$

$$\frac{dy}{dt}=x.\frac{1}{2\sqrt{t}}+\sqrt{t}.\frac{dx}{dt}$$

$$\frac{dy}{dt}=\frac{x}{2}.\frac{1}{\sqrt{t}}+\sqrt{t}.\frac{(a-x)^2}{2a}$$

$$\frac{dy}{dt}=\frac{x}{2}.\sqrt{\frac{a-x}{a+x}}+\sqrt{\frac{a+x}{a-x}}.\frac{(a-x)^2}{2a}\ \ldots(ii)$$

Applying chain rule on equation (i) and equation (ii).

$$\frac{dy}{dx}=\frac{dy}{dt}.\frac{dt}{dx}$$

$$\frac{dy}{dx}=\left[\frac{x}{2}\frac{\sqrt{a-x}}{\sqrt{a+x}}+\frac{\sqrt{a+x}}{\sqrt{a-x}}\frac{(a-x)^2}{2a}\right]\frac{2a}{(a-x)^2}$$

$$\frac{dy}{dx}=\left[\frac{ax(a-x)+(a+x)(a-x)^2}{2a\sqrt{a+x}\sqrt{a-x}}\right]\frac{2a}{(a-x)^2}$$

$$\frac{dy}{dx}=(a-x)\left[\frac{ax+a^2-x^2}{\sqrt{a+x}\sqrt{a-x}}\right]\frac{1}{(a-x)^2}$$

$$\boxed{\frac{dy}{dx}=\frac{a^2-x^2+ax}{(a+x)^{\frac{1}{2}}(a-x)^{\frac{3}{2}}}}$$

(iv) $y=\left(3x^2-2x+7\right)^6$

Solution:

Let $t=3x^2-2x+7$

$$\frac{dt}{dx}=6x-2 \quad \ldots\text{(i)}$$

Also, $y=t^6 \Rightarrow \frac{dy}{dt}=6t^5$

$$\Rightarrow \frac{dy}{dt}=6\left(3x^2-2x+7\right)^5 \ldots\text{(ii)}$$

Applying chain rule on equation (i) and equation (ii)

$$\frac{dy}{dx}=\frac{dy}{dt}.\frac{dt}{dx}$$

$$=6\left(3x^2-2x+7\right)^5.(6x-2)$$

$$\boxed{\frac{dy}{dx}=6(6x-2)\left(3x^2-2x+7\right)^5}$$

(v) $y=\sqrt{\frac{a^2+x^2}{a^2-x^2}}$

Solution:

Let $t=\frac{a^2+x^2}{a^2-x^2}$

$$\Rightarrow y=\sqrt{t}$$

$$\frac{dy}{dt}=\frac{1}{2\sqrt{t}}$$

$$\frac{dy}{dt}=\frac{1}{2}\sqrt{\frac{a^2-x^2}{a^2+x^2}} \quad \ldots\text{(i)}$$

$$t=\frac{a^2+x^2}{a^2-x^2}$$

$$\frac{dt}{dx}=\frac{\left(a^2-x^2\right)(2x)-\left(a^2+x^2\right)(-2x)}{\left(a^2-x^2\right)^2}$$

$$\frac{dt}{dx}=\frac{\cancel{2a^2}x-2x^3+\cancel{2a^2}x+2x^3}{\left(a^2-x^2\right)^2}$$

$$\frac{dt}{dx}=\frac{4a^2x}{\left(a^2-x^2\right)^2} \quad \ldots\text{(ii)}$$

Applying chain rule on equation (i) and equation (ii)

$$\frac{dy}{dx}=\frac{dy}{dt}.\frac{dt}{dx}$$

$$\frac{dy}{dx}=\frac{1}{2}\sqrt{\frac{a^2-x^2}{a^2+x^2}}.\frac{4a^2x}{\left(a^2-x^2\right)^2}$$

$$\frac{dy}{dx}=\frac{2a^2x}{\sqrt{a^2+x^2}\left(a^2-x^2\right)^{\frac{3}{2}}}$$

Q.2 Find $\frac{dy}{dx}$ if:

(i) $3x+4y+7=0$

Solution:

$$3x+4y+7=0$$

Differentiate w.r.t "x"

$$3+4\frac{dy}{dx}=0$$

$$4\frac{dy}{dx}=-3$$

$$\boxed{\frac{dy}{dx}=\frac{-3}{4}}$$

(ii) $xy+y^2=2$

Solution:

$$xy+y^2=2$$

Differentiate w.r.t "x"

$$x\frac{dy}{dx}+y(1)+2y\frac{dy}{dx}=0$$

$$(x+2y)\frac{dy}{dx}=-y$$

$$\boxed{\frac{dy}{dx}=\frac{-y}{x+2y}}$$

(iii) $x^2-4xy-5y=0$

Solution:

$$x^2-4xy-5y=0$$

Differentiate w.r.t "x"

$$2x-4\left[x\frac{dy}{dx}+y(1)\right]-5\frac{dy}{dx}=0$$

$$2x-4x\frac{dy}{dx}-4y-5\frac{dy}{dx}=0$$

$$(4x+5)\frac{dy}{dx}=2(x-2y)$$

$$\boxed{\frac{dy}{dx}=\frac{2(x-2y)}{4x+5}}$$

(iv) $4x^2+2hxy+by^2+2gx+2fy+c=0$

Solution:

$$4x^2+2hxy+by^2+2gx+2fy+c=0$$

Differentiate w.r.t "x".

$$8x+2h\left[x\frac{dy}{dx}+y\right]+2by\frac{dy}{dx}+2g+2f\frac{dy}{dx}=0$$

$$2hx\frac{dy}{dx}+2by\frac{dy}{dx}+2f\frac{dy}{dx}=-8x-2hy-2g$$

$$\cancel{2}(hx+by+f)\frac{dy}{dx}=-\cancel{2}(4x+hy+g)$$

$$\boxed{\frac{dy}{dx}=-\frac{4x+hy+g}{hx+by+f}}$$

(v) $x\sqrt{1+y}+y\sqrt{1+x}=0$

Solution:

$$x\sqrt{1+y}+y\sqrt{1+x}=0$$

Differentiate w.r.t "x",

$$=\frac{d}{dx}\left(x\sqrt{1+y}+y\sqrt{1+x}\right)=\frac{d}{dx}(0)$$

$\Rightarrow$

$$x\frac{d}{dx}\sqrt{1+y}+\sqrt{1+y}\frac{d}{dx}(x)+y\frac{d}{dx}\sqrt{1+x}+\sqrt{1+x}\frac{dy}{dx}=0$$

$$x\frac{1}{2\sqrt{1+y}}.\frac{dy}{dx}+\sqrt{1+y}(1)+y\frac{1}{2\sqrt{1+x}}+\sqrt{1+x}\frac{dy}{dx}=0$$

$$\left(\frac{x}{2\sqrt{1+y}}+\sqrt{1+x}\right)\frac{dy}{dx}=-\left[\sqrt{1+y}+\frac{y}{2\sqrt{1+x}}\right]$$

$$\left(\frac{x+2\sqrt{(1+x)(1+y)}}{\cancel{2}\sqrt{1+y}}\right)\frac{dy}{dx}=-\left[\frac{2\sqrt{(1+x)(1+y)}+y}{\cancel{2}\sqrt{1+x}}\right]$$

$$\frac{x+2\sqrt{1+x+y+xy}}{\sqrt{1+y}}\frac{dy}{dx}=-\frac{y+2\sqrt{1+x+y+xy}}{\sqrt{1+x}}$$

$$\boxed{\frac{dy}{dx}=-\frac{\left(y+2\sqrt{1+x+y+xy}\right)\left(\sqrt{1+y}\right)}{\left(x+2\sqrt{1+x+y+xy}\right)\left(\sqrt{1+x}\right)}}$$

(vi) $y(x^2-1)=x\sqrt{x^2+4}$

Solution:

$$y(x^2-1)=x\sqrt{x^2+4}$$

Differentiate w.r.t "x",

$$\frac{d}{dx}\left(y(x^2-1)\right)=\frac{d}{dx}\left(x\left(\sqrt{x^2+4}\right)\right)$$

$$y\frac{d}{dx}(x^2-1)+(x^2-1)\frac{d}{dx}(y)=x\frac{d}{dx}\sqrt{x^2+4}+\sqrt{x^2+4}\frac{d}{dx}(x)$$

$$y(2x)+(x^2-1)\frac{dy}{dx}=x\frac{1(2x)}{2\sqrt{x^2+4}}+\sqrt{x^2+4}(1)$$

$$2xy+(x^2-1)\frac{dy}{dx}=\frac{x^2}{\sqrt{x^2+4}}+\sqrt{x^2+4}$$

$$(x^2-1)\frac{dy}{dx}=\frac{x^2+x^2+4}{\sqrt{x^2+4}}-2xy$$

$$(x^2-1)\frac{dy}{dx}=\frac{2x^2+4-2xy\left(\sqrt{x^2+4}\right)}{\sqrt{x^2+4}}$$

$$(x^2-1)\frac{dy}{dx}=\frac{2(x^2+2)-2xy\sqrt{x^2+4}}{\sqrt{x^2+4}}$$

$$\boxed{\frac{dy}{dx}=\frac{2(x^2+2)-2xy\sqrt{x^2+4}}{\sqrt{x^2+4}(x^2-1)}}$$

Q.3 Find $\frac{dy}{dx}$ for the following parametric functions:

(i) $x=\theta+\frac{1}{\theta}$ and $y=\theta+1$

Solution:

$$x=\theta+\frac{1}{\theta}$$

Differentiate w.r.t "θ",

$$\frac{dx}{d\theta}=1-\frac{1}{\theta^2}\Rightarrow\frac{dx}{d\theta}=\frac{\theta^2-1}{\theta^2}\quad\ldots(i)$$

$$y=\theta+1$$

Differentiate w.r.t "θ",

$$\frac{dy}{d\theta}=1\ldots(ii)$$

Applying chain rule on equation (i) and equation (ii)

$$\frac{dy}{dx}=\frac{dy}{d\theta}.\frac{d\theta}{dx}$$

$$\frac{dy}{dx}=1.\frac{\theta^2}{\theta^2-1}$$

$$\boxed{\frac{dy}{dx}=\frac{\theta^2}{\theta^2-1}}$$

(ii) $x=\frac{a\left(1-t^2\right)}{1+t^2},\ y=\frac{2bt}{1+t^2}$

Solution:

$$x=\frac{a\left(1-t^2\right)}{1+t^2}$$

Differentiate w.r.t "t",

$$\frac{dx}{dt}=a\frac{\left(1+t^2\right)(-2t)-\left(1-t^2\right)(2t)}{\left(1+t^2\right)^2}$$

$$=a\left[\frac{-2t-\cancel{2t^3}-2t+\cancel{2t^3}}{\left(1+t^2\right)^2}\right]$$

$$\frac{dx}{dt}=\frac{-4at}{\left(1+t^2\right)^2}\quad\ldots(i)$$

$$y=\frac{2bt}{1+t^2}$$

Differentiate w.r.t "t"

$$\frac{dy}{dt}=\frac{\left(1+t^2\right)(2b)-2bt(2t)}{\left(1+t^2\right)^2}$$

$$=\frac{2b+2bt^2-4bt^2}{\left(1+t^2\right)^2}$$

$$=\frac{2b-2bt^2}{\left(1+t^2\right)^2}$$

$$\frac{dy}{dt}=\frac{2b\left(1-t^2\right)}{\left(1+t^2\right)^2}\quad\ldots(ii)$$

Applying chain rule on equation (i) and equation (ii)

$$\frac{dy}{dx}=\frac{dy}{dt}.\frac{dt}{dx}$$

$$\frac{dy}{dx}=\frac{-2b\left(1-t^2\right)}{\cancel{\left(1+t^2\right)^2}}.\frac{\cancel{\left(1+t^2\right)^2}}{4at}$$

$$\boxed{\frac{dy}{dx}=\frac{-b\left(1-t^2\right)}{2at}}$$

Q.4 Prove that $y\frac{dy}{dx}+x=0$ **if**

$$x=\frac{1-t^2}{1+t^2},\ y=\frac{2t}{1+t^2}$$

Solution:

$$x=\frac{1-t^2}{1+t^2}$$

Differentiate w.r.t "t",

$$\frac{dx}{dt}=\frac{\left(1+t^2\right)(-2t)-\left(1-t^2\right)(2t)}{\left(1+t^2\right)^2}$$

$$=\frac{-2t-\cancel{2t^3}-2t+\cancel{2t^3}}{\left(1+t^2\right)^2}$$

$$\frac{dx}{dt}=\frac{-4t}{\left(1+t^2\right)^2}\quad\ldots(i)$$

$$y=\frac{2t}{1+t^2}$$

Differentiate w.r.t "t",

$$\frac{dy}{dt}=\frac{\left(1+t^2\right)(2)-(2t)(2t)}{\left(1+t^2\right)^2}$$

$$=\frac{2+2t^2-4t^2}{\left(1+t^2\right)^2}$$

$$\frac{dy}{dt}=\frac{2\left(1-t^2\right)}{\left(1+t^2\right)^2}\quad\ldots(ii)$$

Applying chain rule on equation (i) and equation (ii)

$$\frac{dy}{dx}=\frac{dy}{dt}.\frac{dt}{dx}$$

$$\frac{dy}{dx}=\frac{-2\left(1-t^2\right)}{\cancel{\left(1+t^2\right)^2}}.\frac{\cancel{\left(1+t^2\right)^2}}{4t}$$

$$\frac{dy}{dx}=\frac{-\left(1-t^2\right)}{2t}$$

$$\Rightarrow y\frac{dy}{dx}=-\frac{1-t^2}{2t}\times\frac{2t}{1+t^2}$$

$$y\frac{dy}{dx}=-\frac{1-t^2}{1+t^2}$$

$$y\frac{dy}{dx}=-x \qquad \because x=\frac{1-t^2}{1+t^2}$$

$$y\frac{dy}{dx}+x=0$$

Hence the proof.

Q.5 Differentiate

(i) $x^2-\frac{1}{x^2}$ w.r.t. x^4

Solution:

Let $y=x^2-\frac{1}{x^2}$, $t=x^4$

We have to find $\frac{dy}{dt}$

$$y=x^2-\frac{1}{x^2}$$

Differentiate w.r.t "x"

$$\frac{dy}{dx}=2x+\frac{2}{x^3}$$

$$\frac{dy}{dx}=\frac{2x^4+2}{x^3} \ldots(i)$$

Also $t=x^4$

Differentiate w.r.t "x"

$$\frac{dt}{dx}=4x^3 \ldots(ii)$$

Applying chain rule on equation (i) and equation (ii),

$$\frac{dy}{dt}=\frac{dy}{dx}\cdot\frac{dx}{dt}$$

$$\frac{dy}{dt}=\frac{2\left(x^4+1\right)}{x^3}\cdot\frac{1}{4x^3}$$

$$\frac{dy}{dt}=\frac{2\left(x^4+1\right)}{4x^6}$$

$$\boxed{\frac{dy}{dt}=\frac{x^4+1}{2x^6}}$$

(ii) $\left(1+x^2\right)^n$ w.r.t x^2

Solution:

Let $y=\left(1+x^2\right)^n$, $t=x^2$

We have to find $\frac{dy}{dt}$

$$y=\left(1+x^2\right)^n$$

Differentiate w.r.t "x"

$$\frac{dy}{dx}=n\left(1+x^2\right)^{n-1}(2x)$$

$$\frac{dy}{dx}=2nx\left(1+x^2\right)^{n-1} \ldots(i)$$

also $t=x^2$

Differentiate w.r.t "x"

$$\frac{dt}{dx}=2x \ldots(ii)$$

Applying chain rule on equation (i) and equation (ii)

$$\frac{dy}{dt}=\frac{dy}{dx}\cdot\frac{dx}{dt}$$

$$\frac{dy}{dt}=\cancel{2}n\cancel{x}\left(1+x^2\right)^{n-1}\cdot\frac{1}{\cancel{2}\cancel{x}}$$

$$\boxed{\frac{dy}{dt}=n\left(1+x^2\right)^{n-1}}$$

(iii) $\frac{x^2+1}{x^2-1}$ w.r.t $\frac{x-1}{x+1}$

Solution:

Let $y=\frac{x^2+1}{x^2-1}$, $t=\frac{x-1}{x+1}$

We have to find $\frac{dy}{dt}$

$$y=\frac{x^2+1}{x^2-1}$$

Differentiate w.r.t "x"

$$\frac{dy}{dx}=\frac{(x^2-1)(2x)-(x^2+1)(2x)}{(x^2-1)^2}$$

$$\frac{dy}{dx}=\frac{2x^3-2x-2x^3-2x}{(x^2-1)^2}$$

$$\frac{dy}{dx}=\frac{-4x}{(x^2-1)^2}\quad\ldots\text{(i)}$$

also $t=\dfrac{x-1}{x+1}$

Differentiate w.r.t "x"

$$\frac{dt}{dx}=\frac{(x+1)(1)-(x-1)(1)}{(x+1)^2}$$

$$=\frac{x+1-x+1}{(x+1)^2}$$

$$\frac{dt}{dx}=\frac{2}{(x+1)^2}\quad\ldots\text{(ii)}$$

Applying chain rule on equation (i) and equation (ii)

$$\frac{dy}{dt}=\frac{dy}{dx}\cdot\frac{dx}{dt}$$

$$\frac{dy}{dt}=\frac{-4x}{(x^2-1)^2}\frac{(x+1)^2}{2}$$

$$\frac{dy}{dt}=-\frac{2x(x+1)^2}{(x+1)^2(x-1)^2}$$

$$\boxed{\frac{dy}{dt}=\frac{-2x}{(x-1)^2}}$$

(iv) $\dfrac{ax+b}{cx+d}$ w.r.t $\dfrac{ax^2+b}{ax^2+d}$

Solution:

Let $y=\dfrac{ax+b}{cx+d}$, $\quad t=\dfrac{ax^2+b}{ax^2+d}$

we have to find $\dfrac{dy}{dt}$

$$y=\frac{ax+b}{cx+d}$$

Differentiate w.r.t "x"

$$\frac{dy}{dx}=\frac{(cx+d)(a)-(ax+b)(c)}{(cx+d)^2}$$

$$=\frac{acx+ad-acx-bc}{(cx+d)^2}$$

$$\frac{dy}{dx}=\frac{ad-bc}{(cx+d)^2}\quad\ldots\text{(i)}$$

also $t=\dfrac{ax^2+b}{ax^2+d}$

Differentiate w.r.t "x"

$$\frac{dt}{dx}=\frac{(ax^2+d)(2ax)-(ax^2+b)(2ax)}{(ax^2+d)^2}$$

$$\frac{dt}{dx}=\frac{2a^2x^3+2adx-2a^2x^3-2abx}{(ax^2+d)^2}$$

$$\frac{dt}{dx}=\frac{2ax(d-b)}{(ax^2+d)^2}\quad\ldots\text{(ii)}$$

Applying chain rule on equation (i) and equation (ii)

$$\frac{dy}{dt}=\frac{dy}{dx}\cdot\frac{dx}{dt}$$

$$\frac{dy}{dt}=\frac{ad-bc}{(cx+d)^2}\cdot\frac{(ax^2+d)^2}{2ax(d-b)}$$

$$\boxed{\frac{dy}{dt}=\frac{(ad-bc)(ax^2+d)^2}{2ax(d-b)(cx+d)^2}}$$

(v) $\frac{x^2+1}{x^2-1}$ w.r.t x^3

Solution:

Let $y=\frac{x^2+1}{x^2-1}$, $t=x^3$

we have to find $\frac{dy}{dt}$

$$y=\frac{x^2+1}{x^2-1}$$

Differentiate w.r.t "x"

$$\frac{dy}{dx}=\frac{(x^2-1)(2x)-(x^2+1)(2x)}{(x^2-1)^2}$$

$$\frac{dy}{dx}=\frac{\cancel{2x^3}-2x-\cancel{2x^3}-2x}{(x^2-1)^2}$$

$$\frac{dy}{dx}=\frac{-4x}{(x^2-1)^2} \quad \ldots\text{(i)}$$

also $t=x^3$

Differentiate w.r.t "x"

$$\frac{dt}{dx}=3x^2 \quad \ldots\text{(ii)}$$

Applying chain rule on equation (i) and equation (ii)

$$\frac{dy}{dt}=\frac{dy}{dx}.\frac{dx}{dt}$$

$$\frac{dy}{dt}=\frac{-4x}{(x^2-1)^2}.\frac{1}{3x^2}$$

$$\boxed{\frac{dy}{dt}=\frac{-4}{3x(x^2-1)^2}}$$

DERIVATIVES OF TRIGONOMETRIC FUNCTIONS:

(1) $\frac{d}{dx}(\sin x) = \cos x$

(2) $\frac{d}{dx}(\cos x) = -\sin x$

(3) $\frac{d}{dx}(\tan x) = \sec^2 x$

(4) $\frac{d}{dx}(\cot x) = -\text{cosec}^2 x$

(5) $\frac{d}{dx}(\sec x) = \sec x \tan x$

(6) $\frac{d}{dx}(\text{cosec}\, x) = -\text{cosec}\, x \cot x$

(1) **Prove that:** $\frac{d}{dx}(\sin x) = \cos x$

Let $y = \sin x$

$$y + \delta y = \sin(x + \delta x)$$

$$\delta y = \sin(x + \delta x) - \sin x$$

$$= 2\cos\left(\frac{x + \delta x + x}{2}\right)\sin\left(\frac{x + \delta x - x}{2}\right)$$

$$\because \sin\alpha - \sin\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$$

$$= 2\cos\left(x + \frac{\delta x}{2}\right)\sin\left(\frac{\delta x}{2}\right)$$

Dividing both sides by 'δx'

$$\frac{\delta y}{\delta x} = 2\cos\left(x + \frac{\delta x}{2}\right)\frac{\sin\left(\frac{\delta x}{2}\right)}{\delta x}$$

$$= \cos\left(x + \frac{\delta x}{2}\right)\frac{\sin\left(\frac{\delta x}{2}\right)}{\left(\frac{\delta x}{2}\right)}$$

Taking limit as $\delta x \to 0$

$$\lim_{\delta x \to 0}\frac{\delta y}{\delta x} = \lim_{\delta x \to 0}\cos\left(x + \frac{\delta x}{2}\right)\lim_{\delta x \to 0}\left(\frac{\sin\left(\frac{\delta x}{2}\right)}{\left(\frac{\delta x}{2}\right)}\right)$$

$$\because \lim_{\theta \to 0}\frac{\sin\theta}{\theta} = 1$$

$$\frac{dy}{dx} = \cos(x) \times 1$$

$$\frac{d}{dx}(\sin x) = \cos x$$

(2) **Prove that:** $\frac{d}{dx}(\cos x) = -\sin x$

Let $y = \cos x$

$$y + \delta y = \cos(x + \delta x)$$

$$\delta y = \cos(x + \delta x) - \cos x$$

$$\delta y = -2\sin\left(\frac{x + \delta x + x}{2}\right)\sin\left(\frac{x + \delta x - x}{2}\right)$$

$$\because \cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$

$$\delta y = -2\sin\left(x + \frac{\delta x}{2}\right)\sin\left(\frac{\delta x}{2}\right)$$

Dividing both sides by 'δx'

$$\frac{\delta y}{\delta x} = \frac{-2\sin\left(x + \frac{\delta x}{2}\right)\sin\left(\frac{\delta x}{2}\right)}{\delta x}$$

$$= -\sin\left(x + \frac{\delta x}{2}\right)\frac{\sin\left(\frac{\delta x}{2}\right)}{\left(\frac{\delta x}{2}\right)}$$

Taking limit as $\delta x \to 0$

$$\lim_{\delta x \to 0}\frac{\delta y}{\delta x} = -\lim_{\delta x \to 0}\sin\left(x + \frac{\delta x}{2}\right)\left[\lim_{\delta x \to 0}\frac{\sin\left(\frac{\delta x}{2}\right)}{\frac{\delta x}{2}}\right]$$

$$\because \lim_{\theta \to 0}\frac{\sin\theta}{\theta} = 1$$

$$\frac{dy}{dx} = -\sin x \times 1$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

(3) **Prove that:** $\frac{d}{dx}(\tan x) = \sec^2 x$

Let $y = \tan x$

$$y = \frac{\sin x}{\cos x}$$

$$y + \delta y = \frac{\sin(x + \delta x)}{\cos(x + \delta x)}$$

$$\delta y=\frac{\sin(x+\delta x)}{\cos(x+\delta x)}-\frac{\sin x}{\cos x}$$

$$=\frac{\sin(x+\delta x)\cos x-\sin x\cos(x+\delta x)}{\cos(x+\delta x)\cos x}$$

$$\delta y=\frac{\sin(x+\delta x-x)}{\cos(x+\delta x)\cos x}$$

$\because \sin\alpha\cos\beta-\cos\alpha\sin\beta=\sin(\alpha-\beta)$

Dividing both sides by 'δx'

$$\frac{\delta y}{\delta x}=\frac{\sin(\delta x)}{\delta x\cos(x+\delta x)\cos x}$$

Taking limit as $\delta x\to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\left[\lim_{\delta x\to 0}\frac{\sin\delta x}{\delta x}\right]\left[\lim_{\delta x\to 0}\frac{1}{\cos(x+\delta x)\cos x}\right]$$

$$\frac{dy}{dx}=1\times\frac{1}{\cos x.\cos x}$$

$$\frac{d}{dx}(\tan x)=\frac{1}{\cos^2 x}$$

$$\frac{d}{dx}(\tan x)=\sec^2 x$$

(4) Prove that: $\frac{d}{dx}(\cot x)=-\text{cosec}^2 x$

Let $y=\cot x$

$$y=\frac{\cos x}{\sin x}$$

$$y+\delta y=\frac{\cos(x+\delta x)}{\sin(x+\delta x)}$$

$$\delta y=\frac{\cos(x+\delta x)}{\sin(x+\delta x)}-y$$

$$\delta y=\frac{\cos(x+\delta x)}{\sin(x+\delta x)}-\frac{\cos x}{\sin x}$$

$$=\frac{\cos(x+\delta x)\sin x-\cos x\sin(x+\delta x)}{\sin(x+\delta x)(\sin x)},$$

$\because \sin\alpha\cos\beta-\cos\alpha\sin\beta=\sin(\alpha-\beta)$

$$=\frac{\sin[x-(x+\delta x)]}{\sin(x+\delta)(\sin x)}$$

$$\delta y=\frac{\sin(-\delta x)}{\sin(x+\delta x)\sin x}$$

$\because \sin(-\theta)=-\sin\theta$

Dividing both sides by 'δx'

$$\frac{\delta y}{\delta x}=-\frac{\sin\delta x}{\delta x[\sin(x+\delta x)\sin x]}$$

Taking limit as $\delta x\to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=-\lim_{\delta x\to 0}\frac{\sin\delta x}{\delta x}.\lim_{\delta x\to 0}\frac{1}{\sin(x+\delta x)\sin x}$$

$$\frac{dy}{dx}=-1\times\frac{1}{\sin x.\sin x}$$

$$\frac{d}{dx}(\cot x)=-\frac{1}{\sin^2 x}$$

$$\frac{d}{dx}(\cot x)=-\text{cosec}^2 x$$

(5) Prove that: $\frac{d}{dx}(\sec x)=\sec x\tan x$

Let $y=\sec x$

$$y+\delta y=\sec(x+\delta x)$$

$$\delta y=\sec(x+\delta x)-y$$

$$\delta y=\sec(x+\delta x)-\sec x$$

$$\delta y=\frac{1}{\cos(x+\delta x)}-\frac{1}{\cos x}$$

$$=\frac{\cos x-\cos(x+\delta x)}{\cos(x+\delta x)\cos x}$$

$\because \cos\alpha-\cos\beta=-2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$

$$\delta y=\frac{-2\sin\left(\frac{x+x+\delta x}{2}\right)\sin\left(\frac{x-(x+\delta x)}{2}\right)}{\cos(x+\delta x)\cos x}$$

Dividing both sides by 'δx'

$$\frac{\delta y}{\delta x}=\frac{-2\sin\left(x+\frac{\delta x}{2}\right)\sin\left(\frac{-\delta x}{2}\right)}{(\delta x)\cos(x+\delta x)\cos x}$$

$$\frac{\delta y}{\delta x}=\frac{\sin\left(x+\frac{\delta x}{2}\right)\sin\left(\frac{\delta x}{2}\right)}{\cos(x+\delta x)\cos x\left(\frac{\delta x}{2}\right)}$$

Taking limit as $\delta x \to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}\sin\left(x+\frac{\delta x}{2}\right)\lim_{\delta x\to 0}\frac{\sin\left(\frac{\delta x}{2}\right)}{\frac{\delta x}{2}}\lim_{\delta x\to 0}\frac{1}{\cos(x+\delta x)\cos x}$$

$$\frac{dy}{dx}=\sin x\times 1\times\frac{1}{\cos x\cos x}$$

$$\frac{d}{dx}(\sec x)=\sec x\tan x$$

(6) Prove that:

$$\frac{d}{dx}(\operatorname{cosec} x)=-\operatorname{cosec} x\cot x$$

Let $y=\operatorname{cosec} x$

$$y=\frac{1}{\sin x}$$

$$y+\delta y=\frac{1}{\sin(x+\delta x)}$$

$$\delta y=\frac{1}{\sin(x+\delta x)}-y$$

$$\delta y=\frac{1}{\sin(x+\delta x)}-\frac{1}{\sin x}$$

$$=\frac{\sin x-\sin(x+\delta x)}{\sin(x+\delta x)\sin x}$$

$$\because \sin\alpha-\sin\beta=2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$$

$$=\frac{2\cos\left(\frac{x+x+\delta x}{2}\right)\sin\left(\frac{x-(x+\delta x)}{2}\right)}{\sin(x+\delta x)\sin x}$$

$$\delta y=\frac{2\cos\left(x+\frac{\delta x}{2}\right)\sin\left(-\frac{\delta x}{2}\right)}{\sin(x+\delta x)\sin x}$$

Dividing both sides by 'δx'

$$\frac{\delta y}{\delta x}=\frac{-2\cos\left(x+\frac{\delta x}{2}\right)\sin\left(\frac{\delta x}{2}\right)}{\delta x\sin(x+\delta x)\sin x}$$

$$\frac{\delta y}{\delta x}=\frac{-\cos\left(x+\frac{\delta x}{2}\right)}{\sin(x+\delta x)\sin x}\cdot\frac{\sin\left(\frac{\delta x}{2}\right)}{\left(\frac{\delta x}{2}\right)}$$

Taking limit as $\delta x \to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=-\lim_{\delta x\to 0}\cos\left(x+\frac{\delta x}{2}\right).\lim_{\delta x\to 0}\frac{1}{\sin(x+\delta x)\sin x}\times\lim_{\delta x\to 0}\left(\frac{\sin\frac{\delta x}{2}}{\frac{\delta x}{2}}\right)$$

$$\frac{dy}{dx}=-\cos x\times\frac{1}{\sin x.\sin x}\times 1$$

$$\frac{d}{dx}(\operatorname{cosec} x)=-\operatorname{cosec} x\cot x$$

DERIVATIVES OF INVERSE TRIGONOMETRIC FUNCTIONS

(1) $\frac{d}{dx}(\sin^{-1}x)=\frac{1}{\sqrt{1-x^2}},\quad -1<x<1$

(2) $\frac{d}{dx}(\cos^{-1}x)=\frac{-1}{\sqrt{1-x^2}},\quad -1<x<1$

(3) $\frac{d}{dx}(\tan^{-1}x)=\frac{1}{1+x^2},\quad -\infty<x<\infty$

(4) $\frac{d}{dx}(\cot^{-1}x)=\frac{-1}{1+x^2},\quad -\infty<x<\infty$

(5) $\frac{d}{dx}(\sec^{-1}x)=\frac{1}{|x|\sqrt{x^2-1}},\quad x\in[-1,1]'$

(6) $\frac{d}{dx}(\operatorname{cosec}^{-1}x)=\frac{-1}{|x|\sqrt{x^2-1}},\quad x\in[-1,1]'$

(1) Prove that:

$$\frac{d}{dx}(\sin^{-1}x)=\frac{1}{\sqrt{1-x^2}},\qquad -1<x<1$$

Let $y=\sin^{-1}x$

$\sin y=x$ (i)

Differentiate w.r.t 'x'

$$\frac{d}{dx}(\sin y)=\frac{d}{dx}(x)$$

$$\cos y\frac{dy}{dx}=1$$

$$\frac{dy}{dx}=\frac{1}{\cos y}$$

$$\frac{dy}{dx}=\frac{1}{\sqrt{\cos^2 y}}$$

$$=\frac{1}{\sqrt{1-\sin^2 y}}$$

Using (i)

$$\frac{d}{dx}(\sin^{-1}x)=\frac{1}{\sqrt{1-x^2}}$$

(2) Prove that:

$$\frac{d}{dx}(\cos^{-1}x)=\frac{-1}{\sqrt{1-x^2}},\quad -1<x<1$$

Let $y=\cos^{-1}x$

$\cos y=x$ (i)

Differentiate w.r.t 'x'

$$\frac{d}{dx}(\cos y)=\frac{d}{dx}(x)$$

$$-\sin y\frac{dy}{dx}=1$$

$$\frac{dy}{dx}=\frac{-1}{\sin y}$$

$$\frac{dy}{dx}=\frac{-1}{\sqrt{\sin^2 y}}$$

$$\frac{d}{dx}(\cos^{-1}x)=\frac{-1}{\sqrt{1-\cos^2 y}}$$

$$\frac{d}{dx}(\cos^{-1}x)=\frac{-1}{\sqrt{1-x^2}}\text{ Using (i)}$$

(3) Prove that:

$$\frac{d}{dx}(\tan^{-1}x)=\frac{1}{1+x^2},\quad -\infty<x<\infty$$

Let $y=\tan^{-1}x$

$\tan y=x$ (i)

Differentiate w.r.t 'x'

$$\frac{d}{dx}(\tan y)=\frac{d}{dx}(x)$$

$$\sec^2 y\frac{dy}{dx}=1$$

$$\frac{dy}{dx}=\frac{1}{\sec^2 y}$$

$$\frac{dy}{dx}=\frac{1}{1+\tan^2 y}$$

Using (i)

$$\frac{d}{dx}(\tan^{-1}x)=\frac{1}{1+x^2}$$

(4) Prove that:

$$\frac{d}{dx}(\cot^{-1}x)=\frac{-1}{1+x^2},\quad -\infty<x<\infty$$

Let $y=\cot^{-1}x$

$\cot y=x$ (i)

Differentiate w.r.t 'x'

$$\frac{d}{dx}(\cot y)=\frac{d}{dx}(x)$$

$$-\text{cosec}^2 y\frac{dy}{dx}=1$$

$$\frac{dy}{dx}=\frac{-1}{\text{cosec}^2 y}$$

$$\frac{dy}{dx}=\frac{-1}{1+\cot^2 y}$$

Using (i)

$$\frac{dy}{dx}=-\frac{1}{1+x^2}$$

$$\frac{d}{dx}(\cot^{-1}y)=-\frac{1}{1+x^2}$$

(5) Prove that:

$$\frac{d}{dx}(\sec^{-1}x)=\frac{1}{|x|\sqrt{x^2-1}},\quad x\in[-1,1]'$$

Let $y=\sec^{-1}x$

$\sec y=x$ (i)

Differentiate w.r.t 'x'

$$\frac{d}{dx}(\sec y)=\frac{d}{dx}(x)$$

$$\sec y\tan y\frac{dy}{dx}=1$$

$$\frac{dy}{dx}=\frac{1}{\sec y\tan y}$$

$$\frac{dy}{dx}=\frac{1}{\sec y\sqrt{\tan^2 y}}$$

$$\frac{dy}{dx}=\frac{1}{\sec y\sqrt{\sec^2 y-1}}$$

Using (i)

$$\frac{dy}{dx}=\frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}\left(\sec^{-1}x\right)=\frac{1}{x\sqrt{x^2-1}}$$

(6) Prove that:

$$\frac{d}{dx}(\text{cosec}^{-1}x)=\frac{-1}{|x|\sqrt{x^2-1}},\quad x\in[-1,1]'$$

Let $y=\text{cosec}^{-1}x$

$\text{cosec}\, y=x$ (i)

Differentiate w.r.t 'x'

$$\frac{d}{dx}(\text{cosec}\, y)=\frac{d}{dx}(x)$$

$$-\text{cosec}\, y\cot y\frac{dy}{dx}=1$$

$$\frac{dy}{dx}=\frac{1}{-\text{cosec}\, y\ \cot y}$$

$$\frac{dy}{dx}=-\frac{1}{\cos ecy\sqrt{\cot^2 y}}$$

$$\frac{dy}{dx}=\frac{1}{-\text{cosec}\, y\sqrt{\text{cosec}^2 y-1}}$$

Using (i)

$$\frac{d}{dx}(\text{cosec}^{-1}x)=-\frac{1}{x\sqrt{x^2-1}}$$

$$\frac{dy}{dx}=-\frac{1}{x\sqrt{x^2-1}}$$