EXERCISE 2.5

Q.1 Differentiate the following trigonometric functions from the first principle.

(i) $\sin 2x$

Solution:

Let $y = \sin 2x$

Taking increments both sides,

$$y + \delta y = \sin 2(x + \delta x)$$

$$\delta y = \sin(2x + 2\delta x) - \sin 2x$$

$$\delta y = 2\cos\left(\frac{2x + 2\delta x + 2x}{2}\right)\sin\left(\frac{\cancel{2x} + 2\delta x - \cancel{2x}}{2}\right)$$

$$\delta y = 2\cos(2x + \delta x).\sin\delta x$$

Dividing both sides by 'δx'

$$\frac{\delta y}{\delta x} = 2\cos(2x + \delta x).\frac{\sin\delta x}{\delta x}$$

Taking limit as $\delta x \to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\left[2\cos(2x + \delta x).\frac{\sin\delta x}{\delta x}\right]$$

$$\frac{dy}{dx} = 2\cos(2x + 0).1 \qquad \left(\because \lim_{\delta x\to 0}\frac{\sin\delta x}{\delta x} = 1\right)$$

$$\boxed{\frac{dy}{dx} = 2\cos 2x}$$

(ii) $\tan 3x$

Solution:

Let $y = \tan 3x$

Taking increments both sides

$$y + \delta y = \tan 3(x + \delta x)$$

$$\delta y = \tan(3x + 3\delta x) - \tan 3x$$

$$\delta y = \frac{\sin(3x + 3\delta x)}{\cos(3x + 3\delta x)} - \frac{\sin 3x}{\cos 3x}$$

$$\delta y = \frac{\cos 3x \sin(3x + 3\delta x) - \sin 3x \cos(3x + 3\delta x)}{\cos 3x.\cos(3x + 3\delta x)}$$

$$\delta y = \frac{\sin(\cancel{3x} + 3\delta x - \cancel{3x})}{\cos 3x.\cos(3x + 3\delta x)}$$

$$\delta y = \frac{\sin 3\delta x}{\cos 3x \cos(3x + 3\delta x)}$$

Dividing both sides by 'δx'

$$\frac{\delta y}{\delta x}=\frac{1}{\cos(3x+3\delta x)\cos 3x}.\frac{3\sin(3\delta x)}{3\delta x}$$

Taking limit as $\delta x\to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}\frac{3}{\cos(3x+3\delta x).\cos 3x}.\lim_{\delta x\to 0}\frac{\sin 3\delta x}{3\delta x}=\frac{3}{\cos 3x\cos 3x}$$

$$\boxed{\frac{dy}{dx}=3\sec^2 3x}$$

(iii) $\sin 2x+\cos 2x$

Solution:

Let $y=\sin 2x+\cos 2x$

Taking increments both sides,

$$y+\delta y=\sin(2x+2\delta x)+\cos(2x+2\delta x)$$

$$\delta y=\sin(2x+2\delta x)+\cos(2x+2\delta x)-\sin 2x-\cos 2x$$

$$\delta y=\left[\sin(2x+2\delta x)-\sin 2x\right]+\left[\cos(2x+2\delta x)-\cos 2x\right]$$

$$\delta y=2\cos\left(\frac{2x+2\delta x+2x}{2}\right)\sin\left(\frac{2x+2\delta x-2x}{2}\right)+\left[-2\sin\left(\frac{2x+2\delta x+2x}{2}\right)\sin\left(\frac{2x+2\delta x-2x}{2}\right)\right]$$

$$\delta y=\left[2\cos(2x+\delta x)\sin\delta x\right]-\left[2\sin(2x+\delta x)\sin\delta x\right]$$

Dividing both sides by 'δx'

$$\frac{\delta y}{\delta x}=2\cos(2x+\delta x).\frac{\sin\delta x}{\delta x}-2\sin(2x+\delta x).\frac{\sin\delta x}{\delta x}$$

Taking limit as $\delta x\to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=2\lim_{\delta x\to 0}\cos(2x+\delta x)\lim_{\delta x\to 0}\frac{\sin\delta x}{\delta x}-2\lim_{\delta x\to 0}\sin(2x+\delta x)\lim_{\delta x\to 0}\frac{\sin\delta x}{\delta x}$$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=2\cos(2x+0).1-2\sin(2x+0).1$$

$$\boxed{\frac{dy}{dx}=2\cos 2x-2\sin 2x}$$

(iv) $\cos x^2$

Solution:

Let $y=\cos x^2$

Taking increments both sides,

$$y+\delta y=\cos(x+\delta x)^2$$

$$\delta y=\cos(x+\delta x)^2-\cos x^2$$

$$\delta y=-2\sin\left(\frac{(x+\delta x)^2+x^2}{2}\right)\sin\left(\frac{(x+\delta x)^2-x^2}{2}\right)$$

$$= -2\sin\left(x^2 + \frac{\delta x^2}{2} + x\delta x\right)\sin\left(\frac{\delta x^2 + 2x\delta x}{2}\right)$$

Dividing both sides by 'δx'

$$= -2\sin\left(x^2 + \frac{\delta x^2}{2} + x\delta x\right)\sin\frac{\left(\frac{\delta x(\delta x + 2x)}{2}\right)}{\left(\frac{\delta x(\delta x + 2x)}{2}\right)} \cdot \frac{(\delta x + 2x)}{2}$$

Taking limit as $\delta x \to 0$

$$\lim_{\delta x \to 0}\frac{\delta y}{\delta x} = -2\lim_{\delta x \to 0}\sin\left(x^2 + \frac{\delta x^2}{2} + x\delta x\right)\lim_{\delta x \to 0}\sin\frac{\left(\frac{\delta x(\delta x + 2x)}{2}\right)}{\left(\frac{\delta x(\delta x + 2x)}{2}\right)}\lim_{\delta x \to 0}\frac{(\delta x + 2x)}{2}$$

$$\lim_{\delta x \to 0}\frac{\delta y}{\delta x} = -2x\sin\left(x^2 + 0\right).1$$

$$\boxed{\frac{dy}{dx} = -2x\sin x^2}$$

(v) $\tan^2 x$

Solution:

Let $y = \tan^2 x$

Taking increments on both sides,

$$y + \delta y = \tan^2(x + \delta x)$$

$$\delta y = \tan^2(x + \delta x) - y$$

$$\delta y = \left[\tan(x + \delta x)\right]^2 - (\tan x)^2$$

$$\delta y = \left[\tan(x + \delta x) + \tan x\right]\left[\tan(x + \delta x) - \tan x\right]$$

$$\delta y = \left[\tan(x + \delta x) + \tan x\right]\left[\frac{\sin(x + \delta x)}{\cos(x + \delta x)} - \frac{\sin x}{\cos x}\right]$$

$$\delta y = \left[\tan(x + \delta x) + \tan x\right]\left[\frac{\sin(x + \delta x)\cos x - \cos(x + \delta x)\sin x}{\cos(x + \delta x)\cos x}\right]$$

$$\delta y = \left[\tan(x + \delta x) + \tan x\right]\left[\frac{\sin(x + \delta x - x)}{\cos(x + \delta x)\cos x}\right]$$

$$\delta y = \left[\tan(x + \delta x) + \tan x\right]\left[\frac{\sin \delta x}{\cos(x + \delta x)\cos x}\right]$$

Dividing both sides by 'δx'

$$\frac{\delta y}{\delta x} = \left[\tan(x + \delta x) + \tan x\right]\left(\frac{1}{\cos(x + \delta x)\cos x}\right) \cdot \frac{\sin \delta x}{\delta x}$$

Taking limit as $\delta x \to 0$

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0}\left[\tan(x+\delta x)+\tan x\right]\left(\lim_{\delta x \to 0} \frac{1}{\cos(x+\delta x)\cos x}\right)\lim_{\delta x \to 0} \frac{\sin \delta x}{\delta x}$$

$$\frac{dy}{dx} = 2\tan x.\frac{1}{\cos^2 x}(1)$$

$$\boxed{\frac{dy}{dx} = 2\tan x \sec^2 x}$$

(vi) $\sqrt{\tan x}$

Solution:

Let $y = \sqrt{\tan x}$

Taking increments on both sides,

$$y+\delta y = \sqrt{\tan(x+\delta x)}$$

$$\delta y = \sqrt{\tan(x+\delta x)} - y$$

$$\delta y = \sqrt{\tan(x+\delta x)} - \sqrt{\tan x}$$

$$\delta y = \left[\sqrt{\tan(x+\delta x)} - \sqrt{\tan x}\right].\left[\frac{\sqrt{\tan(x+\delta x)} + \sqrt{\tan x}}{\sqrt{\tan(x+\delta x)} + \sqrt{\tan x}}\right]$$

$$\delta y = \frac{\tan(x+\delta x) - \tan x}{\sqrt{\tan(x+\delta x)} + \sqrt{\tan x}}$$

Dividing both sides by 'δx'

$$\frac{\delta y}{\delta x} = \frac{1}{\sqrt{\tan(x+\delta x)} + \sqrt{\tan x}}.\left[\frac{1}{\delta x}\frac{\sin(x+\delta x)}{\cos(x+\delta x)} - \frac{\sin x}{\cos x}\right]$$

$$\frac{\delta y}{\delta x} = \frac{1}{\sqrt{\tan(x+\delta x)} + \sqrt{\tan x}}.\frac{1}{\delta x}\left[\frac{\sin(x+\delta x)\cos x - \cos(x+\delta x)\sin x}{\cos(x+\delta x)\cos x}\right]$$

$$\frac{\delta y}{\delta x} = \frac{1}{\sqrt{\tan(x+\delta x)} + \sqrt{\tan x}}.\frac{1}{\delta x}\left[\frac{\sin(x+\delta x - x)}{\cos(x+\delta x)\cos x}\right]$$

$$\frac{\delta y}{\delta x} = \frac{1}{\sqrt{\tan(x+\delta x)} + \sqrt{\tan x}}.\frac{1}{\delta x}\left[\frac{\sin \delta x}{\cos(x+\delta x)\cos x}\right]$$

Taking limit as $\delta x \to 0$

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{1}{\sqrt{\tan(x+\delta x)} + \sqrt{\tan x}} \lim_{\delta x \to 0} \frac{1}{\delta x}\left[\frac{\sin \delta x}{\cos(x+\delta x)\cos x}\right]$$

$$\frac{dy}{dx}=\frac{1}{2\sqrt{\tan x}}\cdot\frac{1}{\cos^2 x}$$

$$\boxed{\frac{dy}{dx}=\frac{\sec^2 x}{2\sqrt{\tan x}}}$$

(vii) $\cos\sqrt{x}$

Solution:

Let $y=\cos\sqrt{x}$

Taking increments on both sides,

$$y+\delta y=\cos\sqrt{x+\delta x}$$

$$\delta y=\cos\sqrt{x+\delta x}-y$$

$$\delta y=\cos\sqrt{x+\delta x}-\cos\sqrt{x}$$

$$\delta y=-2\sin\frac{\sqrt{x+\delta x}+\sqrt{x}}{2}\sin\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}$$

Dividing both sides by 'δx'

$$\frac{\delta y}{\delta x}=-2\sin\frac{\left(\sqrt{x+\delta x}+\sqrt{x}\right)}{2}\cdot\frac{\sin\left(\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}\right)}{\delta x}$$

$$\begin{aligned}\because \delta x&=x+\delta x-x\\&=\left(\sqrt{x+\delta x}\right)^2-\left(\sqrt{x}\right)^2\\&=\left(\sqrt{x+\delta x}+\sqrt{x}\right)\left(\sqrt{x+\delta x}-\sqrt{x}\right)\end{aligned}$$

$$\frac{\delta y}{\delta x}=-\sin\frac{\left(\frac{\sqrt{x+\delta x}+\sqrt{x}}{2}\right)}{\sqrt{x+\delta x}+\sqrt{x}}\frac{\sin\left(\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}\right)}{\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}}$$

Taking limit as $\delta x\to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=-\lim_{\delta x\to 0}\sin\frac{\left(\frac{\sqrt{x+\delta x}+\sqrt{x}}{2}\right)}{\sqrt{x+\delta x}+\sqrt{x}}\lim_{\delta x\to 0}\frac{\sin\left(\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}\right)}{\frac{\sqrt{x+\delta x}-\sqrt{x}}{2}}$$

$$\frac{dy}{dx}=\frac{-\sin\left(\frac{2\sqrt{x}}{2}\right)}{2\sqrt{x}}$$

$$\boxed{\frac{dy}{dx}=-\frac{\sin\sqrt{x}}{2\sqrt{x}}}$$

Q.2 Differentiate the following w.r.t. the variable involved.

(i) $x^2\sec 4x$

Solution:

Let $y=x^2\sec 4x$

Differentiate w.r.t "x"

$$\frac{dy}{dx}=\frac{d}{dx}\left(x^2\sec 4x\right)$$

$$=x^2\frac{d}{dx}(\sec 4x)+\sec 4x\frac{d}{dx}\left(x^2\right)$$

$$=x^2\left[\sec 4x\tan 4x.4\right]+\sec 4x(2x)$$

$$=4x^2\sec 4x\tan 4x+2x\sec 4x$$

$$\boxed{\frac{dy}{dx}=2x\sec 4x(2x\tan 4x+1)}$$

(ii) $\tan^3\theta\sec^2\theta$

Solution:

Let $y=\tan^3\theta\sec^2\theta$

Differentiate w.r.t "θ"

$$\frac{dy}{d\theta}=\frac{d}{d\theta}\left(\tan^3\theta\sec^2\theta\right)$$

$$=\tan^3\theta\frac{d}{d\theta}\left(\sec^2\theta\right)+\sec^2\theta\frac{d}{d\theta}\left(\tan^3\theta\right)$$

$$=\tan^3\theta.\left[2\sec\theta.(\sec\theta\tan\theta)\right]\ +\sec^2\theta\left[3\tan^2\theta.\sec^2\theta\right]$$

$$=2\sec^2\theta\tan^4\theta+3\tan^2\theta\sec^4\theta$$

$$\boxed{\frac{dy}{d\theta}=\sec^2\theta\tan^2\theta\left(2\tan^2\theta+3\sec^2\theta\right)}$$

(iii) $(\sin 2\theta-\cos 3\theta)^2$

Solution:

Let $y=(\sin 2\theta-\cos 3\theta)^2$

Differentiate w.r.t "θ"

$$\frac{dy}{d\theta}=\frac{d}{d\theta}(\sin 2\theta-\cos 3\theta)^2$$

$$=2(\sin 2\theta-\cos 3\theta)\frac{d}{d\theta}(\sin 2\theta-\cos 3\theta)$$

$$\boxed{\frac{dy}{d\theta}=2(\sin 2\theta-\cos 3\theta)[2\cos 2\theta+3\sin 3\theta]}$$

(iv) $\cos\sqrt{x}+\sqrt{\sin x}$

Solution:

Let $y=\cos\sqrt{x}+\sqrt{\sin x}$

Differentiate w.r.t "x"

$$\frac{dy}{dx}=\frac{d}{dx}\left(\cos\sqrt{x}+\sqrt{\sin x}\right)$$

$$=-\sin\sqrt{x}\,\frac{1}{2\sqrt{x}}+\frac{(\cos x)}{2\sqrt{\sin x}}$$

$$\boxed{\frac{dy}{dx}=\frac{-\sin\sqrt{x}}{2\sqrt{x}}+\frac{\cos x}{2\sqrt{\sin x}}}$$

Q.3 Find $\frac{dy}{dx}$ if

(i) $y=x\cos y$

Solution:

$y=x\cos y$

Differentiate w.r.t "x"

$$\frac{dy}{dx}=\frac{d}{dx}(x\cos y)$$

$$\frac{dy}{dx}=x\left(-\sin y\frac{dy}{dx}\right)+\cos y(1)$$

$$\frac{dy}{dx}+x\sin y\frac{dy}{dx}=\cos y$$

$$(1+x\sin y)\frac{dy}{dx}=\cos y$$

$$\boxed{\frac{dy}{dx}=\frac{\cos y}{1+x\sin y}}$$

(ii) $x=y\sin y$

Solution:

$x=y\sin y$

Differentiate w.r.t "x"

$$\frac{d}{dx}(x)=\frac{d}{dx}(y\sin y)$$

$$1=y\cos y\frac{dy}{dx}+\sin y\frac{dy}{dx}$$

$$(y\cos y+\sin y)\frac{dy}{dx}=1$$

$$\boxed{\frac{dy}{dx}=\frac{1}{y\cos y+\sin y}}$$

Q.4 Find the derivative w.r.t "x",

(i) $\cos\sqrt{\frac{1+x}{1+2x}}$

Solution:

Let $y=\cos\sqrt{\frac{1+x}{1+2x}}$

Differentiate w.r.t "x"

$$\frac{dy}{dx}=\frac{d}{dx}\cos\sqrt{\frac{1+x}{1+2x}}$$

$$=-\sin\sqrt{\frac{1+x}{1+2x}}\frac{d}{dx}\sqrt{\frac{1+x}{1+2x}}$$

$$=-\sin\sqrt{\frac{1+x}{1+2x}}.\frac{1}{2}\sqrt{\frac{1+2x}{1+x}}\frac{d}{dx}\left(\frac{1+x}{1+2x}\right)$$

$$=-\sin\sqrt{\frac{1+x}{1+2x}}.\frac{1}{2}\sqrt{\frac{1+2x}{1+x}}\left[\frac{(1+2x)(1)-(1+x)(2)}{(1+2x)^2}\right]$$

$$=-\sin\sqrt{\frac{1+x}{1+2x}}.\frac{1}{2}\sqrt{\frac{1+2x}{1+x}}\left[\frac{(1+2x-2-2x)}{(1+2x)^2}\right]$$

$$\boxed{\frac{dy}{dx}=\frac{\sin\sqrt{\frac{1+x}{1+2x}}}{2\sqrt{1+x}(1+2x)^{\frac{3}{2}}}}$$

(ii) $\sin\sqrt{\frac{1+2x}{1+x}}$

Solution:

Let $y=\sin\sqrt{\frac{1+2x}{1+x}}$

Differentiate w.r.t "x"

$$\frac{dy}{dx}=\frac{d}{dx}\left(\sin\sqrt{\frac{1+2x}{1+x}}\right)$$

$$=\cos\sqrt{\frac{1+2x}{1+x}}.\frac{d}{dx}\sqrt{\frac{1+2x}{1+x}}$$

$$=\cos\sqrt{\frac{1+2x}{1+x}}.\frac{1}{2}\sqrt{\frac{1+x}{1+2x}}.\frac{d}{dx}\left(\frac{1+2x}{1+x}\right)$$

$$=\cos\sqrt{\frac{1+2x}{1+x}}.\frac{1}{2}\sqrt{\frac{1+x}{1+2x}}.\left[\frac{(1+x)(2)-(1+2x)(1)}{(1+x)^2}\right]$$

$$\frac{dy}{dx}=\cos\sqrt{\frac{1+2x}{1+x}}.\frac{1}{2}\sqrt{\frac{1+x}{1+2x}}.\left[\frac{2+2x-1-2x}{(1+x)^2}\right]$$

$$\boxed{\frac{dy}{dx}=\frac{\cos\sqrt{\frac{1+2x}{1+x}}}{2\sqrt{1+2x}.(1+x)^{\frac{3}{2}}}}$$

Q.5 Differentiate

(i) $\sin x$ w.r.t $\cot x$

Solution:

Let $y=\sin x$, and $t=\cot x$

We have to find $\frac{dy}{dt}$

$y=\sin x$

Differentiate w.r.t "x"

$$\frac{dy}{dx}=\cos x\ \ldots\text{(i)}$$

also $t=\cot x$

Differentiate w.r.t "x"

$$\frac{dt}{dx}=-\cos ec^2x\ \ldots\text{(ii)}$$

Applying chain rule on equation (i) and (ii)

$$\frac{dy}{dt}=\frac{dy}{dx}.\frac{dx}{dt}$$

$$\frac{dy}{dt}=\cos x.\left(\frac{-1}{\cos ec^2x}\right)$$

$$\boxed{\frac{dy}{dt}=-\cos x\sin^2 x}$$

(ii) $\sin^2 x$ w.r.t $\cos^4 x$

Solution:

Let $y=\sin^2 x$ and $t=\cos^4 x$

We have to find $\frac{dy}{dt}$

$y=\sin^2 x$

Differentiate w.r.t "x"

$\frac{dy}{dx}=2\sin x\cos x\ldots$(i)

also $t=\cos^4 x$

Differentiate w.r.t "x"

$\frac{dt}{dx}=4\cos^3 x(-\sin x)$

$\frac{dt}{dx}=-4\sin x\cos^3 x\ldots$(ii)

Applying chain rule on equation (i) and (ii)

$\frac{dy}{dt}=\frac{dy}{dx}.\frac{dx}{dt}$

$\frac{dy}{dt}=2\sin x\cos x.\frac{-1}{4\sin x\cos^3 x}$

$$\boxed{\frac{dy}{dt}=-\frac{1}{2}\sec^2 x}$$

Q.6 If $\tan y(1+\tan x)=1-\tan x$**, Show that** $\frac{dy}{dx}=-1$

Solution:

$\tan y(1+\tan x)=1-\tan x$

$\Rightarrow \tan y=\frac{1-\tan x}{1+\tan x}$

$\tan y=\frac{\tan\frac{\pi}{4}-\tan x}{1+\tan\frac{\pi}{4}.\tan x}$ $\qquad \because \tan\frac{\pi}{4}=1$

$\tan y=\tan\left(\frac{\pi}{4}-x\right)$ $\qquad \because \tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$

$\Rightarrow y=\frac{\pi}{4}-x$

Differentiate w.r.t "x"

$$\boxed{\frac{dy}{dx}=-1}$$

Alternate method:

$\Rightarrow \tan y=\frac{1-\tan x}{1+\tan x}$

Differentiate w.r.t "x"

$$\sec^2 y\frac{dy}{dx}=\frac{(1+\tan x)(-\sec^2 x)-(1-\tan x)(\sec^2 x)}{(1+\tan x)^2}$$

$$(1+\tan^2 y)\frac{dy}{dx}=\frac{-\sec^2 x-\sec^2 x\tan x-\sec^2 x+\sec^2 x\tan x}{(1+\tan x)^2}$$

$$\left(1+\left(\frac{1-\tan x}{1+\tan x}\right)^2\right)\frac{dy}{dx}=\frac{-2\sec^2 x}{(1+\tan x)^2}$$

$$\left(\frac{(1+\tan x)^2+(1-\tan x)^2}{(1+\tan x)^2}\right)\frac{dy}{dx}=\frac{-2\sec^2 x}{(1+\tan x)^2}$$

$$\left(\frac{1+2\tan x+\tan^2 x+1-2\tan x+\tan^2 x}{(1+\tan x)^2}\right)\frac{dy}{dx}=\frac{-2\sec^2 x}{(1+\tan x)^2}$$

$$2(1+\tan^2 x)\frac{dy}{dx}=-2\sec^2 x$$

$$2\sec^2 x\frac{dy}{dx}=-2\sec^2 x$$

$$\boxed{\frac{dy}{dx}=-1}$$

Q.7 **If** $y=\sqrt{\tan x+\sqrt{\tan x+\sqrt{\tan x+...\infty}}}$**, Prove that** $(2y-1)\frac{dy}{dx}=\sec^2 x$

Solution:

$$y=\sqrt{\tan x+\sqrt{\tan x+\sqrt{\tan x+...\infty}}}$$

Squaring both sides,

$$y^2=\tan x+\sqrt{\tan x+\sqrt{\tan+...\infty}}$$

$$y^2=\tan x+y$$

Differentiate w.r.t "x"

$$2y\frac{dy}{dx}=\sec^2 x+\frac{dy}{dx}$$

$$2y\frac{dy}{dx}-\frac{dy}{dx}=\sec^2 x$$

$$\boxed{(2y-1)\frac{dy}{dx}=\sec^2 x}$$

Hence proved

Q.8 **If** $x=a\cos^3\theta, y=b\sin^3\theta$**, Show that** $a\frac{dy}{dx}+b\tan\theta=0$

Solution:

$x = a\cos^3\theta$

Differentiate w.r.t "θ"

$$\frac{dx}{d\theta} = -a\left[3\cos^2\theta\sin\theta\right]$$

$$\frac{dx}{d\theta} = -3a\cos^2\theta\sin\theta \ldots\text{(i)}$$

and $y = b\sin^3\theta$

Differentiate w.r.t "θ"

$$\frac{dy}{d\theta} = b\left[3\sin^2\theta\cos\theta\right]$$

$$\frac{dy}{d\theta} = 3b\sin^2\theta\cos\theta \ldots\text{(ii)}$$

Applying chain rule on equation (i) and (ii)

$$\frac{dy}{dx} = \frac{dy}{d\theta}\cdot\frac{d\theta}{dx}$$

$$\frac{dy}{dx} = \cancel{3}b\sin^2\theta\cos\theta.\frac{-1}{\cancel{3}a\cos^2\theta\sin\theta}$$

$$\frac{dy}{dx} = \frac{-b\sin\theta}{a\cos\theta}$$

$$a\frac{dy}{dx} = -b\tan\theta$$

$$\boxed{a\frac{dy}{dx} + b\tan\theta = 0}$$

Hence proved

Q.9 Find $\frac{dy}{dx}$ **if** $x = a(\cos t + \sin t), \quad y = a(\sin t - t\cos t)$

Solution:

$x = a\cos t + a\sin t$

Differentiate w.r.t "t"

$$\frac{dx}{dt} = -a\sin t + a\cos t$$

$$\frac{dx}{dt} = a(\cos t - \sin t) \ldots\text{(i)}$$

And $y = a\sin t - at\cos t$

Differentiate w.r.t "t"

$$\frac{dy}{dt} = a\cos t - a\left[-t\sin t + \cos t\right]$$

$$\frac{dy}{dt} = \cancel{a\cos t} + at\sin t - \cancel{a\cos t}$$

$$\frac{dy}{dt} = at\sin t \ldots\text{(ii)}$$

Applying chain rule on equation (i) and (ii)

$$\frac{dy}{dx}=\frac{dy}{dt}.\frac{dt}{dx}$$

$$\frac{dy}{dx}=at\sin t\frac{1}{a(\cos t-\sin t)}$$

$$\boxed{\frac{dy}{dx}=\frac{t\sin t}{\cos t-\sin t}}$$

Q.10 Differentiate w.r.t. "*x*",

(i) $\cos^{-1}\left(\frac{x}{a}\right)$

Solution:

Let $y=\cos^{-1}\left(\frac{x}{a}\right)$

Differentiate w.r.t "*x*"

$$\frac{dy}{dx}=\frac{d}{dx}\cos^{-1}\left(\frac{x}{a}\right)$$

$$\frac{dy}{dx}=\frac{-1}{\sqrt{1-\left(\frac{x}{a}\right)^2}}\ \frac{d}{dx}\left(\frac{x}{a}\right)$$

$$=\frac{-1}{\sqrt{1-\frac{x^2}{a^2}}}\frac{1}{a}$$

$$=\frac{-1}{\frac{\sqrt{a^2-x^2}}{a}}\frac{1}{a}$$

$$\boxed{\frac{dy}{dx}=\frac{-1}{\sqrt{a^2-x^2}}}$$

(ii) $\cot^{-1}\left(\frac{x}{a}\right)$

Solution:

Let $y=\cot^{-1}\left(\frac{x}{a}\right)$

Differentiate w.r.t "*x*"

$$\frac{dy}{dx}=\frac{-1}{1+\left(\frac{x}{a}\right)^2}\ \frac{d}{dx}\left(\frac{x}{a}\right)$$

$$= -\frac{1}{\frac{a^2+x^2}{a^2}}\cdot\frac{1}{a}$$

$$= \frac{-a^2}{a^2+x^2}\cdot\frac{1}{a}$$

$$\boxed{\frac{dy}{dx} = -\frac{a}{a^2+x^2}}$$

(iii) $\frac{1}{a}\sin^{-1}\left(\frac{a}{x}\right)$

Solution:

Let $y = \frac{1}{a}\sin^{-1}\left(\frac{a}{x}\right)$

Differentiate w.r.t "x"

$$\frac{dy}{dx} = \frac{1}{a}\frac{1}{\sqrt{1-\left(\frac{a}{x}\right)^2}}\left(\frac{-a}{x^2}\right)$$

$$\frac{dy}{dx} = \frac{-1}{\not{a}}\frac{\not{x}}{\sqrt{x^2-a^2}}\left(\frac{\not{a}}{x^{\not{2}}}\right)$$

$$\boxed{\frac{dy}{dx} = \frac{-1}{x\sqrt{x^2-a^2}}}$$

(iv) $\sin^{-1}\sqrt{1-x^2}$

Solution:

Let $y = \sin^{-1}\sqrt{1-x^2}$

Differentiate w.r.t "x"

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-\left(\sqrt{1-x^2}\right)^2}}\cdot\frac{d}{dx}\sqrt{1-x^2}$$

$$= \frac{1}{\sqrt{1-1+x^2}}\frac{1(-2x)}{2\sqrt{1-x^2}}$$

$$= \frac{-1}{\not{x}}\frac{\not{x}}{\sqrt{1-x^2}}$$

$$\boxed{\frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}}}$$

(v) $\sec^{-1}\left(\frac{x^2+1}{x^2-1}\right)$

Solution:

Let $y=\sec^{-1}\left(\frac{x^2+1}{x^2-1}\right)$

Differentiate w.r.t "x"

$$\frac{dy}{dx}=\frac{1}{\left(\frac{x^2+1}{x^2-1}\right)\sqrt{\left(\frac{x^2+1}{x^2-1}\right)^2-1}}\frac{d}{dx}\left(\frac{x^2+1}{x^2-1}\right)$$

$$=\frac{x^2-1}{\left(x^2+1\right)\sqrt{\frac{x^4+2x^2+1-x^4+2x^2-1}{\left(x^2-1\right)^2}}}\left[\frac{\left(x^2-1\right)(2x)-\left(x^2+1\right)(2x)}{\left(x^2-1\right)^2}\right]$$

$$=\frac{x^2-1}{\left(x^2+1\right)\frac{2x}{x^2-1}}\cdot\left[\frac{2x^3-2x-2x^3-2x}{\left(x^2-1\right)^2}\right]$$

$$\frac{dy}{dx}=\frac{-4x}{2x\left(x^2+1\right)}$$

$$\boxed{\frac{dy}{dx}=\frac{-2}{x^2+1}}$$

(vi) $\cot^{-1}\left(\frac{2x}{1-x^2}\right)$

Solution:

Let $y=\cot^{-1}\left(\frac{2x}{1-x^2}\right)$

Differentiate w.r.t "x"

$$\frac{dy}{dx}=\frac{-1}{1+\left(\frac{2x}{1-x^2}\right)^2}\cdot\frac{d}{dx}\left(\frac{2x}{1-x^2}\right)$$

$$=\frac{-1}{1+\frac{4x^2}{1+x^4-2x^2}}\cdot\left[\frac{\left(1-x^2\right)(2)-(2x)(-2x)}{\left(1-x^2\right)^2}\right]$$

$$=\frac{-1}{\frac{1+x^4-2x^2+4x^2}{1+x^4-2x^2}}\cdot\left[\frac{2-2x^2+4x^2}{\left(1-x^2\right)^2}\right]$$

$$=-\frac{\left(1+x^4-2x^2\right)}{1+x^4+2x^2}\cdot\frac{2+2x^2}{\left(1-x^2\right)^2}$$

$$= -\frac{(1-x^2)^2}{(1+x^2)^2} \cdot \frac{2(1+x^2)}{(1-x^2)^2}$$

$$\boxed{\frac{dy}{dx} = -\frac{2}{1+x^2}}$$

(vii) $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

Solution:

Let $y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

Differentiate w.r.t "x"

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-\left(\frac{1-x^2}{1+x^2}\right)^2}} \frac{d}{dx}\left(\frac{1-x^2}{1+x^2}\right)$$

$$= \frac{-1}{\sqrt{1-\frac{1+x^4-2x^2}{1+x^4+2x^2}}} \left[\frac{(1+x^2)(-2x)-(1-x^2)(2x)}{(1+x^2)^2}\right]$$

$$\frac{dy}{dx} = -\frac{(1+x^2)}{\sqrt{4x^2}} \cdot \left[\frac{-2x-2x^3-2x+2x^3}{(1+x^2)^2}\right]$$

$$= \frac{-1}{2x} \cdot \frac{-4x}{1+x^2}$$

$$\boxed{\frac{dy}{dx} = \frac{2}{1+x^2}}$$

Q.11 Show that $\frac{dy}{dx} = \frac{y}{x}$, If $\frac{y}{x} = \tan^{-1}\left(\frac{x}{y}\right)$

Solution:

$$\frac{y}{x} = \tan^{-1}\left(\frac{x}{y}\right)$$

$$y = x\tan^{-1}\frac{x}{y}$$

Differentiate w.r.t "x"

$$\frac{dy}{dx} = x\frac{d}{dx}\tan^{-1}\frac{x}{y} + \tan^{-1}\frac{x}{y}\frac{d}{dx}(x) \qquad \because \tan^{-1}\frac{x}{y} = \frac{y}{x}$$

$$= x\frac{1}{1+\frac{x^2}{y^2}}\frac{d}{dx}\left(\frac{x}{y}\right) + \frac{y}{x}(1)$$

$$= \frac{x}{\frac{x^2+y^2}{y^2}}\left(\frac{y(1)-x\frac{dy}{dx}}{y^2}\right)+\frac{y}{x}$$

$$\frac{dy}{dx}=\frac{xy}{x^2+y^2}-\frac{x^2}{x^2+y^2}\frac{dy}{dx}+\frac{y}{x}$$

$$\frac{dy}{dx}+\frac{x^2}{x^2+y^2}\frac{dy}{dx}=\frac{xy}{x^2+y^2}+\frac{y}{x}$$

$$\frac{dy}{dx}\left(1+\frac{x^2}{x^2+y^2}\right)=\frac{x^2y+y\left(x^2+y^2\right)}{x\left(x^2+y^2\right)}$$

$$\frac{dy}{dx}\left(\frac{x^2+y^2+x^2}{x^2+y^2}\right)=\frac{y\left(x^2+x^2+y^2\right)}{x\left(x^2+y^2\right)}$$

$$\boxed{\frac{dy}{dx}=\frac{y}{x}}$$

Q.12 If $y=\tan\left(p\tan^{-1}x\right)$, **show that** $\left(1+x^2\right)y_1-p\left(1+y^2\right)=0$

Solution:

$$y=\tan\left(p\tan^{-1}x\right)$$

$$\Rightarrow \tan^{-1}y=p\tan^{-1}x$$

Differentiate w.r.t "x",

$$\frac{d}{dx}\left(\tan^{-1}y\right)=p\frac{d}{dx}\left(\tan^{-1}x\right)$$

$$\frac{1}{1+y^2}\frac{dy}{dx}=p\frac{1}{1+x^2}$$

$$\frac{1}{1+y^2}\left(y_1\right)=\frac{p}{1+x^2} \qquad \left(\because \frac{dy}{dx}=y_1\right)$$

$$\left(1+x^2\right)y_1=p\left(1+y^2\right)$$

$$\left(1+x^2\right)y_1-p\left(1+y^2\right)=0$$

Hence proved.

DERIVATIVE OF EXPONENTIAL FUNCTION:

(1) Let $y = e^x$

$$y+\delta y = e^{x+\delta x}$$
$$\delta y = e^{x+\delta x} - y$$
$$\delta y = e^x . e^{\delta x} - e^x$$
$$\delta y = e^x (e^{\delta x} - 1)$$
$$\frac{\delta y}{\delta x} = e^x \left(\frac{e^{\delta x} - 1}{\delta x} \right)$$
$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = e^x \left(\lim_{\delta x \to 0} \frac{e^{\delta x} - 1}{\delta x} \right)$$
$$\frac{dy}{dx} = e^x \times \ln e$$
$$= e^x \times 1$$
$$\boxed{\frac{d}{dx}\left(e^x\right) = e^x}$$

(2) Let $y = a^x$

$$y+\delta y = a^{x+\delta x}$$
$$\delta y = a^{x+\delta x} - y$$
$$\delta y = a^x . a^{\delta x} - a^x$$
$$= a^x \left(a^{\delta x} - 1\right)$$
$$\frac{\delta y}{\delta x} = a^x \left(\frac{a^{\delta x} - 1}{\delta x} \right)$$
$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = a^x \left(\lim_{\delta x \to 0} \frac{a^{\delta x} - 1}{\delta x} \right) \qquad \because \lim_{\delta x \to 0} \frac{a^{\delta x} - 1}{\delta x} = \ell na$$
$$\frac{dy}{dx} = a^x . \ln a$$
$$\boxed{\frac{d}{dx}\left(a^x\right) = a^x \ln a}$$

DERIVATIVE OF THE LOGARITHMIC FUNCTION:

(3) Let $y = \ln x$

$$y + \delta y = \ln(x + \delta x)$$
$$\delta y = \ln(x + \delta x) - y$$
$$\delta y = \ln\left(x + \delta x\right) - \ln x$$

$$\delta y = \ln\left(\frac{x+\delta x}{x}\right) \qquad \because \ell og m - \ell og n = \ell og\frac{m}{n}$$

$$\delta y = \ln\left(1+\frac{\delta x}{x}\right)$$

$$\frac{\delta y}{\delta x} = \frac{1}{\delta x}\ln\left(1+\frac{\delta x}{x}\right)$$

$$\frac{\delta y}{\delta x} = \frac{1}{x}.\frac{x}{\delta x}\ln\left(1+\frac{\delta x}{x}\right)$$

$$\frac{\delta y}{\delta x} = \frac{1}{x}\ln\left(1+\frac{\delta x}{x}\right)^{\frac{x}{\delta x}}$$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \frac{1}{x}\ln\left[\lim_{\delta x\to 0}\left(1+\frac{\delta x}{x}\right)^{\frac{x}{\delta x}}\right] \qquad \because \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n} = e$$

$$\frac{dy}{dx} = \frac{1}{x}\ln e \qquad \because \ell n e = 1$$

$$\boxed{\frac{d}{dx}(\ln x) = \frac{1}{x}}$$

(4) **Let** $y = \log_a x$

$$y+\delta y = \log_a(x+\delta x)$$

$$\delta y = \log_a(x+\delta x) - y$$

$$\delta y = \log_a(x+\delta x) - \log_a x$$

$$\delta y = \log_a\left(\frac{x+\delta x}{x}\right)$$

$$\frac{\delta y}{\delta x} = \frac{1}{\delta x}\log_a\left(1+\frac{\delta x}{x}\right)$$

$$\frac{\delta y}{\delta x} = \frac{1}{x}.\frac{x}{\delta x}\log_a\left(1+\frac{\delta x}{x}\right)$$

$$\frac{\delta y}{\delta x} = \frac{1}{x}.\log_a\left(1+\frac{\delta x}{x}\right)^{\frac{x}{\delta x}}$$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \frac{1}{x}\log_a\left[\lim_{\delta x\to 0}\left(1+\frac{\delta x}{x}\right)^{\frac{x}{\delta x}}\right]$$

$$\frac{dy}{dx} = \frac{1}{x}\log_a e$$

$$\frac{dy}{dx}=\frac{1}{x}\frac{1}{\log_e a}=\frac{1}{x\ln a}$$

$$\boxed{\frac{d}{dx}(\log_a x)=\frac{1}{x\ln a}}$$

DERIVATIVE OF HYPERBOLIC FUNCTIONS:

(1) $\frac{d}{dx}(\sinh x)=\cosh x$

(2) $\frac{d}{dx}(\cosh x)=\sinh x$

(3) $\frac{d}{dx}(\tanh x)=\text{sech}^2 x$

(4) $\frac{d}{dx}(\coth x)=-\text{cosech}^2 x$

(5) $\frac{d}{dx}(\text{sech}\, x)=-\text{sech}\, x\tanh x$

(6) $\frac{d}{dx}(\text{cosech}\, x)=-\text{cosech}\, x\coth x$

Proof:

(1) **Let** $y=\sinh x$

$$y=\frac{e^x-e^{-x}}{2}$$

Differentiate w.r.t 'x'

$$\frac{dy}{dx}=\frac{1}{2}\frac{d}{dx}\left(e^x-e^{-x}\right)$$

$$=\frac{1}{2}\left[\frac{d}{dx}\left(e^x\right)-\frac{d}{dx}\left(e^{-x}\right)\right]$$

$$=\frac{1}{2}\left(e^x+e^{-x}\right)$$

$$\boxed{\frac{d}{dx}(\sinh x)=\cosh x}$$

(2) **Let** $y=\cosh x$

$$y=\frac{e^x+e^{-x}}{2}$$

Differentiate w.r.t 'x'

$$\frac{dy}{dx}=\frac{1}{2}\left[\frac{d}{dx}\left(e^x+e^{-x}\right)\right]$$

$$=\frac{1}{2}\left[\frac{d}{dx}\left(e^x\right)+\frac{d}{dx}\left(e^{-x}\right)\right]$$

$$=\frac{1}{2}\left(e^x-e^{-x}\right)$$

$$\boxed{\frac{d}{dx}(\cosh x)=\sinh x}$$

(3) **Let** $y = \tanh x$

$$y = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Differentiate w.r.t 'x'

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right)$$

$$= \frac{\left(e^x + e^{-x}\right)\frac{d}{dx}\left(e^x - e^{-x}\right) - \left(e^x - e^{-x}\right)\frac{d}{dx}\left(e^x + e^{-x}\right)}{\left(e^x + e^{-x}\right)^2} = \frac{\left(e^x + e^{-x}\right)\left(e^x + e^{-x}\right) - \left(e^x - e^{x}\right)\left(e^x - e^{-x}\right)}{\left(e^x + e^{-x}\right)^2}$$

$$= \frac{\left(e^x + e^{-x}\right)^2 - \left(e^x - e^{-x}\right)^2}{\left(e^x + e^{-x}\right)^2}$$

$$= \frac{e^{2x} + e^{-2x} + 2 - \left(e^{2x} + e^{-2x} - 2\right)}{\left(e^x + e^{-x}\right)^2}$$

$$= \frac{\cancel{e}^{2x} + \cancel{e}^{-2x} + 2 - \cancel{e}^{2x} - \cancel{e}^{-2x} + 2}{\left(e^x + e^{-x}\right)^2}$$

$$= \frac{4}{\left(e^x + e^{-x}\right)^2}$$

$$= \left(\frac{2}{e^x + e^{-x}}\right)^2$$

$$\boxed{\frac{d}{dx}\left(\tanh x\right) = \operatorname{sech}^2 x}$$

(4) **Let** $y = \coth x$

$$y = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

Differentiate w.r.t 'x'

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{e^x + e^{-x}}{e^x - e^{-x}}\right)$$

$$= \frac{\left(e^x - e^{-x}\right)\frac{d}{dx}\left(e^x + e^{-x}\right) - \left(e^x + e^{-x}\right)\frac{d}{dx}\left(e^x - e^{-x}\right)}{\left(e^x - e^{-x}\right)^2}$$

$$= \frac{\left(e^x - e^{-x}\right)\left(e^x - e^{-x}\right) - \left(e^x + e^{-x}\right)\left(e^x + e^{-x}\right)}{\left(e^x - e^{-x}\right)^2}$$

$$= \frac{\left(e^x - e^{-x}\right)^2 - \left(e^x + e^{-x}\right)^2}{\left(e^x - e^{-x}\right)^2}$$

$$=\frac{e^{2x}+e^{-2x}-2-\left(e^{2x}+e^{-2x}+2\right)}{\left(e^{x}-e^{-x}\right)^{2}}$$

$$=\frac{e^{2x}+e^{-2x}-2-e^{2x}-e^{-2x}-2}{\left(e^{x}-e^{-x}\right)^{2}}$$

$$=-\frac{4}{\left(e^{x}-e^{-x}\right)^{2}}$$

$$=-\left(\frac{2}{e^{x}-e^{-x}}\right)^{2}$$

$$\boxed{\frac{d}{dx}(\coth x)=-\operatorname{cosech}^{2}x}$$

(5) **Let** $y=\operatorname{sech} x$

$$y=\frac{2}{e^{x}+e^{-x}}$$

Differentiate w.r.t 'x'

$$\frac{dy}{dx}=2\frac{d}{dx}\left(\frac{1}{e^{x}+e^{-x}}\right)$$

$$\frac{dy}{dx}=2\left[\frac{\left(e^{x}+e^{-x}\right)\frac{d}{dx}(1)-1\frac{d}{dx}\left(e^{x}+e^{-x}\right)}{\left(e^{x}+e^{-x}\right)^{2}}\right]$$

$$\frac{dy}{dx}=2\left[\frac{0-\left(e^{x}-e^{-x}\right)}{\left(e^{x}+e^{-x}\right)^{2}}\right]$$

$$=-2\frac{1}{\left(e^{x}+e^{-x}\right)^{2}}\left(e^{x}-e^{-x}\right)$$

$$=-\left(\frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}\right)\frac{2}{\left(e^{x}+e^{-x}\right)}$$

$$\boxed{\frac{d}{dx}(\operatorname{sech} x)=-\tanh x\operatorname{sech} x}$$

(6) **Let** $y=\operatorname{cosech} x$

$$y=\frac{2}{e^{x}-e^{-x}}$$

Differentiate w.r.t 'x'

$$\frac{dy}{dx}=2\frac{d}{dx}\left(e^{x}-e^{-x}\right)^{-1}$$

$$=-2\left(e^{x}-e^{-x}\right)^{-2}\frac{d}{dx}\left(e^{x}-e^{-x}\right)$$

$$= -2\frac{e^x + e^{-x}}{\left(e^x - e^{-x}\right)^2}$$

$$= -\frac{e^x + e^{-x}}{e^x - e^{-x}} \cdot \frac{2}{e^x - e^{-x}}$$

$$\frac{d}{dx}(\operatorname{cosech} x) = -\coth x \operatorname{cosech} x$$

Inverse Hyperbolic Functions

(1) $\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right)$

(2) $\cosh^{-1} x = \ln\left(x + \sqrt{x^2-1}\right)$

(1) **Let** $y = \sinh^{-1} x$ for $x, y \in R$ then

$$\sinh y = x$$

$$x = \sinh y$$

$$x = \frac{e^y - e^{-y}}{2}$$

$$2x = e^y - e^{-y}$$

$$2x = e^y - \frac{1}{e^y}$$

$$2x = \frac{e^{2y} - 1}{e^y}$$

$$2e^y x = e^{2y} - 1$$

$$\left(e^y\right)^2 - 2x\left(e^y\right) - 1 = 0$$

Which is quadratic in e^y. Now we have

$$e^y = \frac{-(-2x) \pm \sqrt{(-2x)^2 - 4(1)(-1)}}{2(1)}$$

$$= \frac{2x \pm \sqrt{4x^2+4}}{2}$$

$$= \frac{2x \pm 2\sqrt{x^2+1}}{2}$$

$$e^y = x \pm \sqrt{x^2+1}$$

As e^y is positive for $y \in R$

So $e^y = x + \sqrt{x^2+1}$

$$\ln\left(e^y\right) = \ln\left(x + \sqrt{x^2+1}\right)$$

$$y = \ln\left(x + \sqrt{x^2+1}\right)$$

$$\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right)$$

(2) **Let** $y=\cosh^{-1}x$ $\left(\text{for } x\in(1,\infty),\ y\in(0,\infty)\right)$

$\cosh y = x$

$x=\cosh y$

$x=\dfrac{e^y+e^{-y}}{2}$

$2x=e^y+e^{-y}$

$2x=e^y+\dfrac{1}{e^y}$

$2x=\dfrac{e^{2y}+1}{e^y}$

$2xe^y=e^{2y}+1$

$e^{2y}-2xe^y+1=0$

$\left(e^y\right)^2-2x\left(e^y\right)+1=0$

Which is quadratic in e^y, then we have

$e^y=\dfrac{-(-2x)\pm\sqrt{(-2x)^2-4(1)(1)}}{2(1)}$

$=\dfrac{2x\pm\sqrt{4x^2-4}}{2}$

$=\dfrac{2x\pm2\sqrt{x^2-1}}{2}$

$=x\pm\sqrt{x^2-1}$

As e^y is positive

So $e^y=x+\sqrt{x^2-1}$

$\ln\left(e^y\right)=\ln\left(x+\sqrt{x^2-1}\right)$

$y=\ln\left(x+\sqrt{x^2-1}\right)$

$\cosh^{-1}x=\ln\left(x+\sqrt{x^2-1}\right)$

DERIVATIVE OF INVERSE HYPERBOLIC FUNCTIONS:

(1) $\dfrac{d}{dx}\left(\sinh^{-1}x\right)=\dfrac{1}{\sqrt{1+x^2}}$

(2) $\dfrac{d}{dx}\left(\cosh^{-1}x\right)=\dfrac{1}{\sqrt{x^2-1}}$

(3) $\dfrac{d}{dx}\left(\tanh^{-1}x\right)=\dfrac{1}{1-x^2}$

(4) $\dfrac{d}{dx}\left(\coth^{-1}x\right)=\dfrac{1}{1-x^2}$

(5) $\frac{d}{dx}\left(\text{sech}^{-1}x\right)=\frac{-1}{x\sqrt{1-x^2}}$

(6) $\frac{d}{dx}\left(\text{cosech}^{-1}x\right)=\frac{-1}{x\sqrt{1+x^2}}$

Proof:

(1) **Let** $y=\sinh^{-1}x$

$\sinh y=x$

Differentiate w.r.t 'x'

$\frac{d}{dx}(\sinh y)=\frac{d}{dx}(x)$

$\cosh y\frac{dy}{dx}=1$

$\frac{dy}{dx}=\frac{1}{\cosh y}$ $\left(\cosh^2 y=1+\sinh^2 y\right)$

$\frac{dy}{dx}=\frac{1}{\sqrt{1+\sinh^2 y}}$

$\boxed{\frac{dy}{dx}=\frac{1}{\sqrt{1+x^2}}}$

(2) **Let** $y=\cosh^{-1}x$

$\cosh y=x$

Differentiate w.r.t 'x'

$\frac{d}{dx}(\cosh y)=\frac{d}{dx}(x)$

$\sinh y\frac{dy}{dx}=1$

$\frac{dy}{dx}=\frac{1}{\sinh y}$

$\frac{dy}{dx}=\frac{1}{\sqrt{\cosh^2 y-1}}$ $\left(\sinh^2 y=\cosh^2 y-1\right)$

$\boxed{\frac{d}{dx}\left(\cosh^{-1}x\right)=\frac{1}{\sqrt{x^2-1}}}$

(3) **Let** $y=\tanh^{-1}x$

$\tanh y=x$

Differentiate w.r.t 'x'

$\frac{d}{dx}(\tanh y)=\frac{d}{dx}(x)$

$\text{sech}^2 y\frac{dy}{dx}=1$

$\frac{dy}{dx}=\frac{1}{\text{sech}^2 y}$ $\because \text{sech}^2 y=1-\tanh^2 y$

$\frac{dy}{dx}=\frac{1}{1-\tanh^2 y}$

$\boxed{\frac{d}{dx}\left(\tanh^{-1}x\right)=\frac{1}{1-x^2}}$

(4) **Let** $y = \coth^{-1} x$

$\coth y = x$

Differentiate w.r.t 'x'

$$\frac{d}{dx}(\coth y) = \frac{d}{dx}(x)$$

$$-\text{cosech}^2 y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{-\text{cosech}^2 y} \qquad \because \text{cosech}^2 y = \coth^2 y - 1$$

$$\frac{dy}{dx} = \frac{-1}{\coth^2 y - 1}$$

$$= \frac{1}{1 - \coth^2 y}$$

$$\boxed{\frac{d}{dx}(\coth^{-1} x) = \frac{1}{1 - x^2}}$$

(5) **Let** $y = \text{sech}^{-1} x$

$\text{sech}\, y = x$

Differentiate w.r.t 'x'

$$\frac{d}{dx}(\text{sech}\, y) = \frac{d}{dx}(x)$$

$$-\text{sech}\, y \tanh y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = -\frac{1}{\text{sech}\, y \tanh y} \qquad \because \tanh^2 y = 1 - \text{sech}^2 y$$

So $\dfrac{dy}{dx} = -\dfrac{1}{\text{sech}\, y\sqrt{1 - \text{sech}^2 y}}$

$$\boxed{\frac{d}{dx}(\text{sech}^{-1} x) = -\frac{1}{x\sqrt{1 - x^2}}}$$

(6) **Let** $y = \text{cosech}^{-1} x$

$\text{cosech}\, y = x$

Differentiate w.r.t 'x'

$$\frac{d}{dx}(\text{cosech}\, y) = \frac{d}{dx}(x)$$

$$-\text{cosech}\, y \coth y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = -\frac{1}{\text{cosech}\, y \coth y} \qquad \because \coth^2 y = 1 + \text{cosech}^2 y$$

$$\frac{d}{dx}(\text{cosech}^{-1} x) = -\frac{1}{\text{cosech}\, y\sqrt{1 + \text{cosech}^2 y}}$$

$$\boxed{\frac{d}{dx}(\text{cosech}^{-1} x) = -\frac{1}{x\sqrt{1 + x^2}}}$$