SUCCESSIVE DIFFERENTIATION (OR HIGHER DERIVATIVES):

We state different notations used for derivatives of higher orders.

1st Derivative	$y', \frac{dy}{dx}, y_1, Dy, f'(x), \frac{df}{dx}$
2nd Derivative	$y'', \frac{d^2y}{dx^2}, y_2, D^2y, f''(x), \frac{d^2f}{dx^2}$
3rd Derivative	$y''', \frac{d^3y}{dx^3}, y_3, D^3y, f'''(x), \frac{d^3f}{dx^3}$
.	. . . . . .
.	. . . . . .
.	. . . . . .
nth Derivative	$y^n, \frac{d^ny}{dx^n}, y_n, D^ny, f^n(x), \frac{d^nf}{dx^n}$

EXERCISE 2.7

Q.1 Find y_2 if

(i) $y = 2x^5 - 3x^4 + 4x^3 + x - 2$

Solution:

$y = 2x^5 - 3x^4 + 4x^3 + x - 2$

Differentiate w.r.t "x"

$\frac{dy}{dx} = \frac{d}{dx}\left(2x^5 - 3x^4 + 4x^3 + x - 2\right)$

$y_1 = 10x^4 - 12x^3 + 12x^2 + 1$

Differentiate again w.r.t "x"

$\boxed{y_2 = 40x^3 - 36x^2 + 24x}$

(ii) $y = (2x+5)^{\frac{3}{2}}$

Solution:

$y = (2x+5)^{\frac{3}{2}}$

Differentiate w.r.t "x"

$y_1 = \frac{3}{2}(2x+5)^{\frac{3}{2}-1}.2$

$y_1 = \frac{3}{\cancel{2}}(2x+5)^{\frac{1}{2}}.\cancel{2}$

$y_1 = 3(2x+5)^{\frac{1}{2}}$

Differentiate again w.r.t "x"

$y_2 = 3\left(\frac{1}{\cancel{2}}\right)(2x+5)^{\frac{-1}{2}}(\cancel{2})$

$\boxed{y_2 = \frac{3}{\sqrt{2x+5}}}$

(iii) $y = \sqrt{x} + \frac{1}{\sqrt{x}}$

Solution:

$y = \sqrt{x} + \frac{1}{\sqrt{x}}$

Differentiate w.r.t "x"

$y_1 = \frac{1}{2\sqrt{x}} + \left(\frac{-1}{2}x^{\frac{-3}{2}}\right)$

$y_1 = \frac{1}{2\sqrt{x}} - \frac{1}{2x\sqrt{x}}$

Differentiate again w.r.t "x"

$y_2 = \frac{1}{2}\left[\frac{-1}{2}x^{\frac{-3}{2}} + \frac{3}{2}x^{\frac{-5}{2}}\right]$

$y_2 = \frac{1}{4}\left[\frac{-1}{x^{\frac{3}{2}}} + \frac{3}{x^{\frac{5}{2}}}\right]$

$\boxed{y_2 = \frac{3-x}{4x^{\frac{5}{2}}}}$

Q.2 Find y_2 if

(i) $y = x^2 e^{-x}$

Solution:

$y = x^2 e^{-x}$

Differentiate w.r.t "x"

$$\frac{dy}{dx} = x^2 \frac{d}{dx}\left(e^{-x}\right) + e^{-x}\frac{d}{dx}\left(x^2\right)$$

$$= x^2 e^{-x}(-1) + e^{-x}(2x)$$

$$y_1 = e^{-x}\left(2x - x^2\right)$$

Differentiate again w.r.t "x"

$$y_2 = e^{-x}\frac{d}{dx}\left(2x - x^2\right) + \left(2x - x^2\right)\frac{d}{dx}\left(e^{-x}\right)$$

$$= e^{-x}(2-2x) + \left(2x - x^2\right)\left(e^{-x}\right)(-1)$$

$$= e^{-x}\left[2 - 2x - 2x + x^2\right]$$

$$\boxed{y_2 = e^{-x}\left(x^2 - 4x + 2\right)}$$

(ii) $y = \ln\left(\frac{2x+3}{3x+2}\right)$

Solution:

$$y = \ln\left(\frac{2x+3}{3x+2}\right)$$

$$y = \ln(2x+3) - \ln(3x+2)$$

Differentiate w.r.t "x"

$$y_1 = \frac{1}{2x+3}(2) - \frac{1}{3x+2}(3)$$

$$y_1 = 2(2x+3)^{-1} - 3(3x+2)^{-1}$$

Differentiate again w.r.t "x"

$$y_2 = 2.(-1)(2x+3)^{-2}(2) - 3(-1)(3x+2)^{-2}(3)$$

$$y_2 = \frac{-4}{(2x+3)^2} + \frac{9}{(3x+2)^2}$$

$$= \frac{9}{9x^2+12x+4} - \frac{4}{4x^2+12x+9}$$

$$= \frac{9\left(4x^2+12x+9\right) - 4\left(9x^2+12x+4\right)}{(2x+3)^2(3x+2)^2}$$

$$= \frac{\cancel{36}x^2 + 108x + 81 - \cancel{36}x^2 - 48x - 16}{(2x+3)^2(3x+2)^2}$$

$$\boxed{y_2 = \frac{60x+65}{(2x+3)^2(3x+2)^2}}$$

Q.3 Find y_2 if,

(i) $x^2 + y^2 = a^2$

Solution:

$x^2 + y^2 = a^2$

Differentiate w.r.t "x"

$$2x + 2yy_1 = 0$$

$$\cancel{2}yy_1 = -\cancel{2}x$$

$$\Rightarrow y_1 = -\frac{x}{y}$$

Differentiate again w.r.t "x"

$$y_2 = -\frac{y(1) - x(y_1)}{y^2}$$

$$y_2 = -\frac{y - x\left(-\frac{x}{y}\right)}{y^2} \qquad \left(\because y_1 = -\frac{x}{y}\right)$$

$$y_2 = -\frac{x^2+y^2}{y^3}$$

$$\boxed{y_2 = -\frac{a^2}{y^3}} \qquad \left(\because x^2 + y^2 = a^2\right)$$

(ii) $x^3 - y^3 = a^3$

Solution:

$x^3 - y^3 = a^3$

Differentiate w.r.t "x"

$$3x^2 - 3y^2 y_1 = 0$$

$$\cancel{3}y^2 y_1 = \cancel{3}x^2$$

$$y_1 = \frac{x^2}{y^2}$$

Differentiate again w.r.t "x"

$$y_2 = \frac{y^2(2x) - x^2(2yy_1)}{\left(y^2\right)^2}$$

$$y_2 = \frac{2xy^2 - 2x^2\cancel{y}\left(\frac{x^2}{y^2}\right)}{y^4}$$

$$y_2=\frac{2xy^2-\frac{2x^4}{y}}{y^4}$$

$$y_2=\frac{2xy^3-2x^4}{y^5}$$

$$y_2=\frac{-2x\left(x^3-y^3\right)}{y^5}$$

$$\boxed{y_2=\frac{-2a^3x}{y^5}}\quad\left(\because x^3-y^3=a^3\right)$$

(iii) $x=a\cos\theta,\qquad y=a\sin\theta$

Solution:

$x=a\cos\theta$

Differentiate w.r.t "θ"

$$\frac{dx}{d\theta}=-a\sin\theta\ldots\text{(i)}$$

$$\frac{dy}{d\theta}=a\cos\theta\ldots\text{(ii)}$$

Applying chain rule on equation (i) and (ii)

$$\frac{dy}{dx}=\frac{dy}{d\theta}.\frac{d\theta}{dx}$$

$$y_1=\cancel{a}\cos\theta.\left(\frac{-1}{\cancel{a}\sin\theta}\right)$$

$$y_1=-\cot\theta$$

Differentiate again w.r.t "x"

$$y_2=-\left(-\operatorname{cosec}^2\theta\right).\frac{d\theta}{dx}$$

$$y_2=\operatorname{cosec}^2\theta.\left(\frac{-1}{a\sin\theta}\right)$$

$$\boxed{y_2=-\frac{1}{a}\operatorname{cosec}^3\theta}$$

(iv) $x=at^2,\qquad y=bt^4$

Solution:

$x=at^2$

Differentiate w.r.t "t"

$$\frac{dx}{dt}=2at\ldots\text{(i)}$$

$y=bt^4$

Differentiate w.r.t "t"

$$\frac{dy}{dt}=4bt^3\ldots\text{(ii)}$$

Applying chain rule on equation (i) and (ii)

$$\frac{dy}{dx}=\frac{dy}{dt}.\frac{dt}{dx}$$

$$y_1=4bt^3.\frac{1}{2at}$$

$$y_1=\frac{2bt^2}{a}$$

Differentiate again w.r.t "x"

$$y_2=\frac{2b}{a}(2t).\frac{dt}{dx}$$

$$=\frac{4bt}{a}\frac{dt}{dx}$$

$$=\frac{4bt}{a}.\frac{1}{2at}$$

$$\boxed{y_2=\frac{2b}{a^2}}$$

(v) $x^2+y^2+2gx+2fy+c=0$

Solution:

$x^2+y^2+2gx+2fy+c=0$

Differentiate w.r.t "x"

$$2x+2yy_1+2g+2fy_1=0$$

$$(2y+2f)y_1=-2x-2g$$

$$(y+f)y_1=-(x+g)$$

$$y_1=-\frac{x+g}{y+f}\ldots\text{(i)}$$

Differentiate again w.r.t "x"

$$y_2=-\frac{(y+f)(1)-(x+g)(y_1)}{(y+f)^2}$$

$$y_2=-\frac{y+f-\left(-\frac{x+g}{y+f}\right)(x+g)}{(y+f)^2}$$

$$y_2=-\frac{(y+f)^2+(x+g)^2}{(y+f)^3}$$

$$y_2=-\frac{y^2+f^2+2fy+x^2+g^2+2gx}{(y+f)^3}$$

$$y_2 = -\frac{x^2+y^2+2gx+2fy+g^2+f^2}{(y+f)^3}$$

$$\boxed{y_2 = \frac{c-f^2-g^2}{(y+f)^3}} \quad \boxed{\because x^2+y^2+2gx+2fy=-c}$$

Q.4 Find y_4 if

(i) $y=\sin 3x$

Solution:

$y=\sin 3x$

Differentiate w.r.t "x"

$y_1=\cos 3x.3 \Rightarrow y_1 = 3\cos 3x$

Differentiate again w.r.t "x"

$y_2 = 3(-\sin 3x.3) \Rightarrow y_2 = -9\sin 3x$

Differentiate again w.r.t "x"

$y_3 = -9\cos 3x.3 \Rightarrow y_3 = -27\cos 3x$

Differentiate again w.r.t "x"

$y_4 = -27(-3\sin 3x)$

$\boxed{y_4 = 81\sin 3x}$

(ii) $y=\cos^3 x$

Solution:

$y=\cos^3 x$

Differentiate w.r.t "x"

$y_1 = 3\cos^2 x(-\sin x)$

$y_1 = -3\sin x\cos^2 x$

$y_1 = -3\sin x(1-\sin^2 x)$

$y_1 = -3\sin x + 3\sin^3 x$

Differentiate again w.r.t "x"

$y_2 = -3\cos x + 9\sin^2 x(\cos x)$

$= -3\cos x + 9\cos x(1-\cos^2 x)$

$= -3\cos x + 9\cos x - 9\cos^3 x$

$y_2 = 6\cos x - 9\cos^3 x \ldots$(i)

$y_2 = 6\cos x - 9y$

Differentiate again w.r.t "x"

$y_3 = -6\sin x - 9y_1$

Differentiate y_3 w.r.t "x"

$y_4 = -6\cos x - 9y_2$

$y_4 = -6\cos x - 9(6\cos x - 9\cos^3 x)$ using (i)

$y_4 = -6\cos x - 54\cos x + 81\cos^3 x$

$\boxed{y_4 = 81\cos^3 x - 60\cos x}$

(iii) $y=\ln(x^2-9)$

Solution:

$y=\ln(x^2-9)$

$=\ln(x-3)(x+3)$

$y=\ln(x-3)+\ln(x+3)$

Differentiate w.r.t "x"

$$y_1 = \frac{1}{x-3}+\frac{1}{x+3}$$

Differentiate again w.r.t "x"

$$y_2 = \frac{-1}{(x-3)^2}+\frac{-1}{(x+3)^2}$$

$$y_2 = -\left[\frac{1}{(x-3)^2}+\frac{1}{(x+3)^2}\right]$$

Differentiate again w.r.t "x"

$$y_3 = -\left[\frac{-2}{(x-3)^3}+\frac{-2}{(x+3)^3}\right]$$

$$y_3 = 2\left[\frac{1}{(x-3)^3}+\frac{1}{(x+3)^3}\right]$$

Differentiate again w.r.t "x"

$$y_4 = 2\left[\frac{-3}{(x-3)^4}+\frac{-3}{(x+3)^4}\right]$$

$$y_4 = \frac{-6}{(x-3)^4}+\frac{-6}{(x+3)^4}$$

$$\boxed{y_4 = -6\left[\frac{1}{(x-3)^4}+\frac{1}{(x+3)^4}\right]}$$

Q.5 If $x=\sin\theta,\ y=\sin m\theta$ show that

$(1-x^2)y_2 - xy_1 + m^2 y = 0$

Solution:

$x=\sin\theta$

$\Rightarrow \theta = \sin^{-1}x$

So $y=\sin m\theta$

Becomes $y=\sin(m\sin^{-1}x)$

Differentiate w.r.t "x"

$$\frac{dy}{dx}=\cos\left(m\sin^{-1}x\right).\frac{d}{dx}\left(m.\sin^{-1}x\right)$$

$$y_1=\cos\left(m\sin^{-1}x\right)m.\frac{1}{\sqrt{1-x^2}}$$

Or $\sqrt{1-x^2}.y_1=m.\cos\left(m\sin^{-1}x\right)$

Differentiate again w.r.t "x"

$$\sqrt{1-x^2}.y_2+y_1.\frac{(-2x)}{2\sqrt{1-x^2}}=m\left[-\sin\left(m\sin^{-1}x\right)\right].\left[\frac{m}{\sqrt{1-x^2}}\right]$$

$$\sqrt{1-x^2}y_2-xy_1.\frac{1}{\sqrt{1-x^2}}=-m^2y.\frac{1}{\sqrt{1-x^2}}\quad\because y=\sin m\sin^{-1}x$$

Multiply both sides by $\sqrt{1-x^2}$

We get $\left(1-x^2\right)y_2-xy_1=-m^2y$

Or $\boxed{\left(1-x^2\right)y_2-xy_1+m^2y=0}$

Q.6 If $y=e^x\sin x$, **show that**

$$\frac{d^2y}{dx^2}-2\frac{dy}{dx}+2y=0$$

Solution:

$$y=e^x\sin x$$

Differentiate w.r.t "x"

$$\frac{dy}{dx}=e^x\frac{d}{dx}(\sin x)+\sin x\frac{d}{dx}\left(e^x\right)$$

$$\frac{dy}{dx}=e^x\cos x+e^x\sin x$$

$$\frac{dy}{dx}=e^x\cos x+y\ \ldots\text{(i)}$$

Differentiate again w.r.t "x"

$$\frac{d^2y}{dx^2}=e^x\left((-\sin x)+\cos x\right)+\frac{dy}{dx}$$

$$\frac{d^2y}{dx^2}=-e^x\sin x+e^x\cos x+\frac{dy}{dx}$$

$$\frac{d^2y}{dx^2}=-y+e^x\cos x+\frac{dy}{dx}$$

Adding and subtracting "$e^x\sin x$"

$$\frac{d^2y}{dx^2}=-y+e^x\sin x+e^x\cos x+\frac{dy}{dx}-e^x\sin x$$

$$\frac{d^2y}{dx^2}=-y+\frac{dy}{dx}-y+\frac{dy}{dx}$$

$$\boxed{\because\frac{dy}{dx}=e^x\cos x+e^x\sin x}$$

$$\frac{d^2y}{dx^2}=2\frac{dy}{dx}-2y$$

$$\frac{d^2y}{dx^2}-2\frac{dy}{dx}+2y=0$$

Hence proved

Q.7 If $y=e^{ax}\sin bx$, **show that**

$$\frac{d^2y}{dx^2}-2a\frac{dy}{dx}+\left(a^2+b^2\right)y=0$$

Solution:

$$y=e^{ax}\sin bx\ \ldots\text{(i)}$$

Differentiate w.r.t "x"

$$\frac{dy}{dx}=e^{ax}.\cos bx.b+\sin bx\left(e^{ax}.a\right)$$

$$\frac{dy}{dx}=ae^{ax}\sin bx+be^{ax}\cos bx$$

$$\frac{dy}{dx}=ay+be^{ax}\cos bx\ \ldots\text{(ii)}\qquad\left(\because e^{ax}\sin bx=y\right)$$

Differentiate again w.r.t "x"

$$\frac{d^2y}{dx^2}=a\frac{dy}{dx}+b\left[e^{ax}\left(-b\sin bx\right)+\cos bx.e^{ax}.a\right]$$

$$=a\frac{dy}{dx}-e^{ax}b^2\sin bx+abe^{ax}\cos bx$$

$$\frac{d^2y}{dx^2}=a\frac{dy}{dx}-b^2e^{ax}\sin bx+abe^{ax}\cos bx$$

Adding and subtracting 'a^2y'

$$\frac{d^2y}{dx^2}=a\frac{dy}{dx}-b^2y+abe^{ax}\cos bx+a^2y-a^2y$$

$$\frac{d^2y}{dx^2}=a\frac{dy}{dx}-b^2y+a\left(be^{ax}\cos bx+ay\right)-a^2y$$

$$\frac{d^2y}{dx^2}=a\frac{dy}{dx}-b^2y+a\left(\frac{dy}{dx}\right)-a^2y$$

$$\frac{d^2y}{dx^2}=a\frac{dy}{dx}-b^2y+a\frac{dy}{dx}-a^2y$$

$$\frac{d^2y}{dx^2}=2a\frac{dy}{dx}-\left(a^2+b^2\right)y$$

$$\Rightarrow\boxed{\frac{d^2y}{dx^2}-2a\frac{dy}{dx}+\left(a^2+b^2\right)y=0}$$

Q.8 If $y=\left(\cos^{-1}x\right)^2$, **show that** $\left(1-x^2\right)y_2-xy_1-2=0$

Solution:

$$y=\left(\cos^{-1}x\right)^2$$

Differentiate w.r.t "x"

$$\frac{dy}{dx}=2\cos^{-1}x\left(\frac{-1}{\sqrt{1-x^2}}\right)$$

$$\sqrt{1-x^2}\,y_1=-2\cos^{-1}x$$

Differentiate again w.r.t "x"

$$\sqrt{1-x^2}\frac{d}{dx}(y_1)+y_1\frac{d}{dx}\sqrt{1-x^2}=-2\left(\frac{-1}{\sqrt{1-x^2}}\right)$$

$$\sqrt{1-x^2}\,y_2-y_1.\frac{1(-2x)}{2\sqrt{1-x^2}}=-2\left(\frac{-1}{\sqrt{1-x^2}}\right)$$

$$\frac{\left(1-x^2\right)y_2-xy_1}{\sqrt{1-x^2}}=\frac{2}{\sqrt{1-x^2}}$$

$$\Rightarrow \boxed{\left(1-x^2\right)y_2-xy_1-2=0}$$

Q.9 If $y=a\cos(\ln x)+b\sin(\ln x)$, **show that** $x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=0$

Solution:

$$y=a\cos(\ln x)+b\sin(\ln x)$$

Differentiate w.r.t "x"

$$\frac{dy}{dx}=\frac{-a\sin(\ln x)}{x}+\frac{b\cos(\ln x)}{x}$$

$$x\frac{dy}{dx}=-a\sin(\ln x)+b\cos(\ln x)$$

Differentiate again w.r.t "x"

$$x\frac{d^2y}{dx^2}+\frac{dy}{dx}(1)=\frac{-a\cos(\ln x)}{x}-\frac{b\sin(\ln x)}{x}$$

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}=-\left[a\cos(\ln x)+b\sin(\ln x)\right]$$

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}=-y$$

$$\Rightarrow \boxed{x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=0}$$

Maclaurin's Theorem:

The power series expansion of a function $f(x)$ is

$f(x)=a_0+a_1x+a_2x^2+a_3x^3+......+a_nx^n+...$ where $a_0,a_1,a_2.........a_n,...$ are constants and x is a variable. Now we find all constants by finding successive derivatives of the power series and evaluating them at $x=0$

$$f(x)=a_0+a_1x+a_2x^2+a_3x^3+......+a_nx^n+... \longrightarrow \textbf{(i)}$$

$$f(0)=a_0 \longrightarrow \textbf{(ii)}$$

$$f'(x)=a_1+2a_2x+3a_3x^2+......+na_nx^{n-1}+...$$

$$f'(0)=a_1 \longrightarrow \textbf{(iii)}$$

$$f''(x)=2a_2+6a_3x+......+n(n-1)a_nx^{n-2}+...$$

$$f''(0)=2a_2,\quad a_2=\frac{1}{2!}f''(0) \longrightarrow \textbf{(iv)}$$

$$f'''(x)=6a_3+......+n(n-1)(n-2)a_nx^{n-3}+...$$

$$f'''(0)=6a_3,\quad a_3=\frac{1}{3!}f'''(0) \longrightarrow \textbf{(v)}$$

Putting (ii), (iii), (iv), (v) in (i)

$$\boxed{f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+......+\frac{x^n}{n!}f^n(0)+...}$$

This expansion is called **Maclaurin series** or **Maclaurin's theorem**

Note:

A function f can be expanded in the Maclaurin series if the function is defined in the interval containing 0 and its derivatives exist at $x=0$. The expansion is valid only if it is convergent.

Tailor's Series Expansions of Functions:

If f is defined in the interval containing $'a'$ and its derivatives of all orders exist at $x=a$ then we can expand $f(x)$ as

$$f(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3+.....+\frac{f^n(a)}{n!}(x-a)^n+...$$

Proof:

Let

$$f(x)=a_o+a_1(x-a)+a_2(x-a)^2+a_3(x-a)^3+a_4(x-a)^4+........+a_n(x-a)^n+... \longrightarrow \textbf{(i)}$$

$$f(a)=a_o \longrightarrow \textbf{(ii)}$$

$$f'(x)=a_1+2a_2(x-a)+3a_3(x-a)^2+4a_4(x-a)^3+....+na_n(x-a)^{n-1}+...$$

$$f'(a)=a_1 \longrightarrow \textbf{(iii)}$$

$$f''(x)=0+2a_2+6a_3(x-a)+12a_4(x-a)^2+....+n(n-1)a_n(x-a)^{n-2}+...$$

$$f''(a)=2a_2$$

$$\frac{1}{2}f''(a)=a_2 \longrightarrow \textbf{(iv)}$$

$$f'''(x)=0+6a_3+24a_4(x-a)+....+n(n-1)(n-2)a_n(x-a)^{n-3}+....$$

$$f'''(a)=6a_3$$

$$\frac{1}{6}f'''(a)=a_3$$

$$\frac{1}{3!}f'''(a)=a_3 \longrightarrow \textbf{(v)}$$

Similarly

$$\frac{1}{n!}f^n(a)=a_n \longrightarrow \textbf{(vi)}$$

Putting (ii), (iii), (iv), (v), (vi) in (i)

$$\boxed{f(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+.....+\frac{f^n(a)}{n!}(x-a)^n+...}$$

This expansion of function $f(x)$ is called **Tailor series** expansion.