EXERCISE 2.8

Q.1 Apply the Maclaurin series expansion to prove that:

(i) $\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+...$

Solution:

Let

$f(x)=\ln(1+x)$ and $f(0)=0$

$f'(x)=\frac{1}{1+x}$ and $f'(0)=1$

$f''(x)=\frac{-1}{(1+x)^2}$ and $f''(0)=-1$

$f'''(x)=\frac{2}{(1+x)^3}$ and $f'''(0)=2$

$f^{iv}(x)=\frac{-6}{(1+x)^4}$ and $f^{iv}(0)=-6$

Applying Maclaurin's series expansion,

$$f(x)=f(0)+\frac{f'(0)x}{\lfloor 1}+\frac{f''(0)x^2}{\lfloor 2}+\frac{f'''(0)x^3}{\lfloor 3}+...$$

$$\ln(1+x)=0+\frac{1}{\lfloor 1}x+\frac{-1}{\lfloor 2}x^2+\frac{2}{\lfloor 3}x^3+...$$

$$\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+...$$

(ii) Prove that $\cos x=1-\frac{x^2}{\lfloor 2}+\frac{x^4}{\lfloor 4}-\frac{x^6}{\lfloor 6}+...$

Solution:

Let $f(x)=\cos x$ and $f(0)=1$

$f'(x)=-\sin x$ and $f'(0)=0$

$f''(x)=-\cos x$ and $f''(0)=-1$

$f'''(x)=\sin x$ and $f'''(0)=0$

$f^{iv}(x)=\cos x$ and $f^{iv}(0)=1$

$f^{v}(x)=-\sin x$ and $f^{v}(0)=0$

$f^{vi}(x)=-\cos x$ and $f^{vi}(0)=-1$

Applying Maclaurin's series expansion,

$$f(x)=f(0)+\frac{f'(0)}{\lfloor 1}x+\frac{f''(0)}{\lfloor 2}x^2+\frac{f'''(0)}{\lfloor 3}x^3+...$$

$$\cos x=1+\frac{0}{\lfloor 1}x+\frac{-1}{\lfloor 2}x^2+\frac{0}{\lfloor 3}x^3+\frac{1}{\lfloor 4}x^4+...$$

$$\cos x=1-\frac{x^2}{\lfloor 2}+\frac{x^4}{\lfloor 4}-\frac{x^6}{\lfloor 6}+...$$

(iii) $\sqrt{1+x}=1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}+...$

Solution:

Let

$f(x)=\sqrt{1+x}$ and $f(0)=1$

$f'(x)=\frac{1}{2\sqrt{1+x}}$ and $f'(0)=\frac{1}{2}$

$f''(x)=\frac{-1}{4(1+x)^{\frac{3}{2}}}$ and $f''(0)=\frac{-1}{4}$

$f'''(x)=\frac{3}{8(1+x)^{\frac{5}{2}}}$ and $f'''(0)=\frac{3}{8}$

$f^{iv}(x)=\frac{-15}{16(1+x)^{\frac{7}{2}}}$ and $f^{iv}(0)=\frac{-15}{16}$

Applying Maclaurin's series expansion,

$$f(x)=f(0)+\frac{f'(0)}{\lfloor 1}x+\frac{f''(0)}{\lfloor 2}x^2+\frac{f'''(0)}{\lfloor 3}x^3+...$$

$$\sqrt{1+x}=1+\frac{1}{2}x+\frac{\frac{-1}{4}}{\lfloor 2}x^2+\frac{\frac{3}{8}}{\lfloor 3}x^3+...$$

$$\sqrt{1+x}=1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}+...$$

(iv) $e^x=1+x+\frac{x^2}{\lfloor 2}+\frac{x^3}{\lfloor 3}+...$

Solution:

Let

$f(x)=e^x$ and $f(0)=1$

$f'(x)=e^x$ and $f'(0)=1$

$f''(x)=e^x$ and $f''(0)=1$

$f'''(x)=e^x$ and $f'''(0)=1$

Applying Maclaurin's series expansion,

$$f(x)=f(0)+\frac{f'(0)}{\lfloor 1}x+\frac{f''(0)}{\lfloor 2}x^2+\frac{f'''(0)}{\lfloor 3}x^3+...$$

$$e^x=1+\frac{1}{\lfloor 1}x+\frac{1}{\lfloor 2}x^2+\frac{1}{\lfloor 3}x^3+...$$

$$\Rightarrow e^x=1+x+\frac{x^2}{\lfloor 2}+\frac{x^3}{\lfloor 3}+...$$

(v) $e^{2x}=1+2x+\frac{4x^2}{\lfloor 2}+\frac{8x^3}{\lfloor 3}+...$

Solution:

Let

$f(x)=e^{2x}$ and $f(0)=1$

$f'(x)=2e^{2x}$ and $f'(0)=2$

$f''(x)=4e^{2x}$ and $f''(0)=4$

$f'''(x)=8e^{2x}$ and $f'''(0)=8$

$f^{iv}(x)=16e^{2x}$ and $f^{iv}(0)=16$

Applying Maclaurin's series expansion,

$$f(x)=f(0)+f'\frac{(0)}{\lfloor 1}x+f''\frac{(0)}{\lfloor 2}x^2+f'''\frac{(0)}{\lfloor 3}x^3+...$$

$$e^{2x}=1+2x+\frac{4x^2}{\lfloor 2}+\frac{8x^3}{\lfloor 3}+...$$

Q.2 Show that $\cos(x+h)=\cos x-h\sin x-\frac{h^2}{\lfloor 2}\cos x+\frac{h^3}{\lfloor 3}\sin x+...$

and evaluate $\cos 61°$**.**

Solution:

Let

$f(x+h)=\cos(x+h)$

$\Rightarrow f(x)=\cos x$

$f'(x)=-\sin x$ and $f''(x)=-\cos x$

$f'''(x)=\sin x$ and $f^{iv}(x)=\cos x$

Applying Tailor's Theorem, we have

$$f(x+h)=f(x)+\frac{h}{\lfloor 1}f'(x)+\frac{h^2}{\lfloor 2}f''(x)+\frac{h^3}{\lfloor 3}f'''(x)+...$$

$$\cos(x+h)=\cos x-h\sin x+\frac{h^2}{\lfloor 2}\cos x-\frac{h^3}{\lfloor 3}\sin x+...$$

For $x=60°$ and $h=1°$ or $h=0.017455$

$$\cos(60°+1°)=\cos 60°-(0.017455)\sin 60°-\frac{(0.017455)^2}{\lfloor 2}\cos 60°+\frac{(0.017455)^3}{\lfloor 3}\sin 60°$$

$$\cos 61^\circ = \frac{1}{2} - \frac{\sqrt{3}}{2}(0.017455) - \frac{1}{4}(0.017455)^2 + \frac{\sqrt{3}}{12}(0.017455)^3$$
$$\approx 0.5 - 0.015116 - 0.0000761 + 0.00000076$$
$$\approx 0.4848$$

Q.3 Show that $2^{x+h} = 2^x\left[1 + (\ln 2)h + \frac{(\ln 2)^2}{\lfloor 2}h^2 + \frac{(\ln 2)^3}{\lfloor 3}h^3 + \ldots\right]$

Solution:

Let $f(x+h) = 2^{x+h}$

$\Rightarrow f(x) = 2^x$

$f'(x) = 2^x(\ln 2)$ and $f''(x) = 2^x(\ln 2)^2$

$f'''(x) = 2^x(\ln 2)^3$ and $f^{iv}(x) = 2^x(\ln 2)^4$

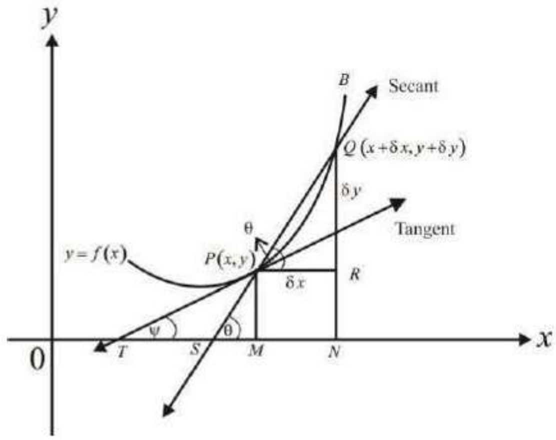
Applying Taylor's Theorem

$$f(x+h) = f(x) + \frac{h}{\lfloor 1} f'(x) + \frac{h^2}{\lfloor 2} f''(x) + \frac{h^3}{\lfloor 3} f'''(x) + \ldots$$

$$2^{x+h} = 2^x + \frac{h}{\lfloor 1}.2^x(\ln 2) + \frac{h^2}{\lfloor 2} 2^x(\ln 2)^2 + \ldots$$

$$2^{x+h} = 2^x\left[1 + (\ln 2)h + (\ln 2)^2\frac{h^2}{\lfloor 2} + (\ln 2)^3\frac{h^3}{\lfloor 3} + \ldots\right]$$

GEOMETRICAL INTERPRETATION OF A DERIVATIVE:



Let $P(x,y)$ and $Q(x+\delta x, y+\delta y)$ be two neighboring points on the graph of the function defined by the equation $y = f(x)$. The line PQ is a secant to the curve. Its inclination is θ. TP is the tangent to the curve at point P. Its inclination is ψ

In ΔPQR

$$\tan\theta = \frac{QR}{PR} = \frac{\delta y}{\delta x}$$

Applying limit $\delta x \to 0$, the secant will become the tangent at P and θ will tend to ψ.

$$\lim_{\delta x \to 0} \tan\theta = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

$$\tan\theta = \frac{dy}{dx}$$

The derivative w.r.t 'x' of the function defined by the equation $y = f(x)$ is equal to the slope of the tangent to the graph of the function at point $P(x,y)$.

INCREASING AND DECREASING FUNCTIONS:

Let f be defined on interval (a,b) and let $x_1, x_2 \in (a,b)$ then

(i) f is increasing on the interval (a,b) if $f(x_2) > f(x_1)$ whenever $x_2 > x_1$

(ii) f is decreasing on the interval (a,b) if $f(x_2) < f(x_1)$ whenever $x_2 > x_1$

Note:

(i) A differentiable function f is increasing on (a,b) if tangent lines to its graph at all points $(x, f(x))$ have positive slopes i.e. $f'(x) > 0$, $\forall x \in (a,b)$.

(ii) A differentiable function f is decreasing on (a,b) if tangent lines to its graph at all points $(x, f(x))$ have negative slopes i.e. $f'(x) < 0$, $\forall x \in (a,b)$.

$f'(x) < 0$ $\forall x$ such that $a < x < b$

Stationary Point:

A point where f is neither increasing nor decreasing is called a stationary point, provided that $f'(x)=0$ at that point.

RELATIVE EXTREMA:

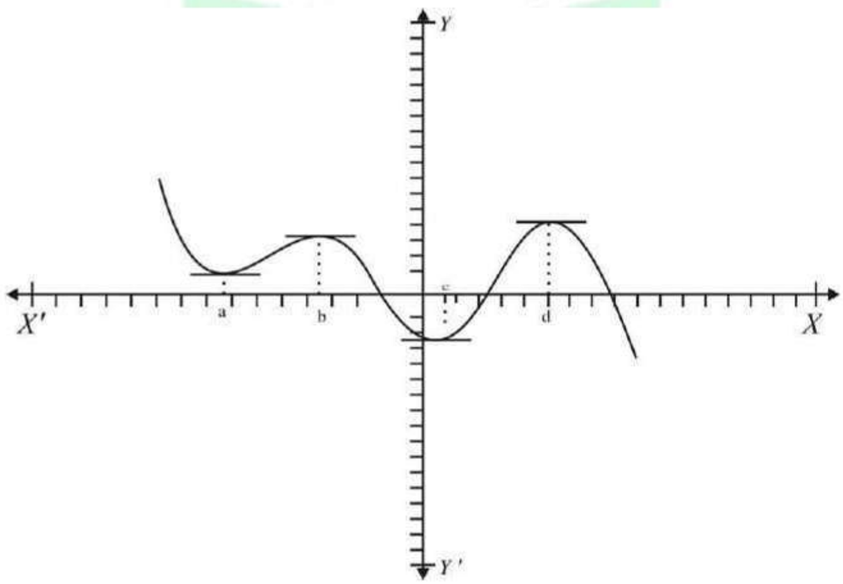
Let $(c-\delta x, c+\delta x)\subseteq D_f$ (domain of a function f) where δx is small positive number

If $f(c)\geq f(x)\ \forall x\in(c-\delta x, c+\delta x)$ then the function f is said to have a relative maxima at $x=c$

If $f(c)\leq f(x)\forall x\in(c-\delta x, c+\delta x)$ then the function f has relative minima at $x=c$.

Both relative maximum and minimum are called relative extrema (in general).

The graph of a function is shown in the adjoining figure. It has relative maxima at $x=b$ and $x=d$. But at $x=a$ and $x=c$ it has relative minima.



Critical Values and Critical Points:

If $c\in D_f$ and $f'(c)=0$ or $f'(c)$ does not exist then the number $f(c)$ is called a critical value of f while the point $(c, f(c))$ on the graph of $f(x)$ is named as a critical point.

There are functions which have extrema (maxima or minima) at the points where their derivatives do not exist.

First derivative rule:

Let $f(x)$ f be differentiable in neighbourhood of c where $f'(c)=0$

(i) If $f'(x)$ changes sign from positive to negative as x increases through c then $f(c)$ is the relative maxima of $f(x)$.

(ii) If $f'(x)$ changes sign from negative to positive as x increases through c then $f(c)$ is the relative minima of $f(x)$.