

Second Derivative Rule:

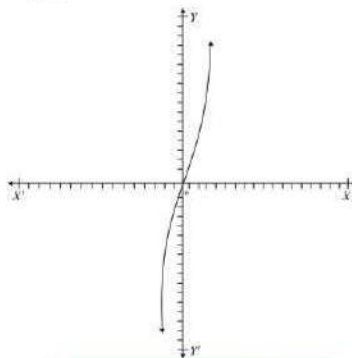
Let $f(x)$ be a differentiable function in neighborhood of c where $f'(c) = 0$ then

- (i) $f(x)$ has relative maxima at c if $f''(c) < 0$
- (ii) $f(x)$ has relative minima at c if $f''(c) > 0$.

Note:

(i) A stationary point is called a turning point if it is either a maximum point or a minimum point.

(ii) If $f'(x) > 0$ before the point $x = a$, $f'(x) = 0$ at $x = a$ and $f'(x) > 0$ after $x = a$ then f does not have a relative maxima. Such a point of the function is called the point of inflection.



EXERCISE 2.9

Q.1 Determine the intervals in which f is increasing or decreasing for the domain mentioned in each case.

(i) $f(x) = \sin x, x \in (-\pi, \pi)$

Solution:

$$f(x) = \sin x$$

Differentiate w.r.t. "x"

$$f'(x) = \cos x$$

$$\text{Put } f'(x) = 0$$

$$\cos x = 0$$

$$\Rightarrow x = -\frac{\pi}{2}, \frac{\pi}{2}$$

Intervals are $\left(-\pi, -\frac{\pi}{2}\right), \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

and $\left(\frac{\pi}{2}, \pi\right)$

$$\text{For } \left(-\pi, -\frac{\pi}{2}\right); f'(x) = \cos x < 0$$

So f is decreasing

$$\text{For } \left(-\frac{\pi}{2}, \frac{\pi}{2}\right); f'(x) = \cos x > 0$$

So f is increasing.

$$\text{For } \left(\frac{\pi}{2}, \pi\right); f'(x) = \cos x < 0$$

So f is decreasing

(ii) $f(x) = \cos x, x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Solution:

$$f(x) = \cos x$$

Differentiate w.r.t. "x"

$$f'(x) = -\sin x$$

$$\text{Put } f'(x) = 0$$

$$-\sin x = 0$$

$$\Rightarrow x = 0$$

Intervals are $\left(-\frac{\pi}{2}, 0\right)$ and $\left(0, \frac{\pi}{2}\right)$

$$\text{For } \left(-\frac{\pi}{2}, 0\right); f'(x) = -\sin x > 0$$

So f is increasing

$$\text{For } \left(0, \frac{\pi}{2}\right); \quad f'(x) = -\sin x < 0$$

So f is decreasing

(iii) $f(x) = 4 - x^2, x \in (-2, 2)$

Solution:

$$f(x) = 4 - x^2$$

Differentiate w.r.t. "x"

$$f'(x) = -2x$$

$$\text{Put } f'(x) = 0$$

$$-2x = 0$$

$$x = 0$$

Intervals are $(-2, 0)$ and $(0, 2)$

$$\text{For } (-2, 0); \quad f'(x) = -2x > 0$$

So f is increasing

$$\text{For } (0, 2); \quad f'(x) = -2x < 0$$

So f is decreasing

(iv) $f(x) = x^2 + 3x + 2, x \in (-4, 1)$

Solution:

$$f(x) = x^2 + 3x + 2$$

Differentiate w.r.t. "x"

$$f'(x) = 2x + 3$$

$$\text{Put } f'(x) = 0$$

$$2x + 3 = 0$$

$$x = -\frac{3}{2}$$

Intervals are $\left(-4, -\frac{3}{2}\right)$ and $\left(-\frac{3}{2}, 1\right)$

$$\text{For } \left(-4, -\frac{3}{2}\right); \quad f'(x) = 2x + 3 < 0$$

So f is decreasing

$$\text{For } \left(-\frac{3}{2}, 1\right); \quad f'(x) = 2x + 3 > 0$$

So f is increasing

Q.2 Find the extreme values for the following functions defined as,

(i) $f(x) = 1 - x^3$

Solution:

$$f(x) = 1 - x^3$$

Differentiate w.r.t. "x"

$$f'(x) = -3x^2$$

$$\text{Put } f'(x) = 0$$

$$-3x^2 = 0 \Rightarrow x = 0$$

$$\text{Now } f''(x) = -6x$$

$$\text{Putting } x = 0,$$

$$f''(0) = 0$$

So the second derivative is not helpful in determining the extreme values.

Now we use first derivative test.

$$\text{Let } x = 0 - \varepsilon$$

$$\text{and } f'(0 - \varepsilon) = -3(0 - \varepsilon)^2 = -3\varepsilon^2 < 0$$

$$\text{and } f'(0 + \varepsilon) = -3(0 + \varepsilon)^2 = -3\varepsilon^2 < 0$$

So the first derivative does not change sign at $x = 0$ and $f(x) = 1$,

$\therefore (0, 1)$ is the point of inflexion.

(ii) $f(x) = x^2 - x - 2$

Solution:

$$f(x) = x^2 - x - 2$$

Differentiate w.r.t. "x"

$$f'(x) = 2x - 1$$

$$\text{Put } f'(x) = 0 \Rightarrow 2x - 1 = 0$$

$$\text{we get } x = \frac{1}{2}$$

$$\text{take } f'(x) = 2x - 1$$

Differentiate again w.r.t. "x"

$$f''(x) = 2$$

$$\text{at } x = \frac{1}{2}$$

$$f''(x) = 2 > 0$$

so f has relative minima at $x = \frac{1}{2}$

$$\text{and } f\left(\frac{1}{2}\right) = \frac{-9}{4}$$

(iii) $f(x) = 5x^2 - 6x + 2$

Solution:

$$f(x) = 5x^2 - 6x + 2$$

Differentiate w.r.t. "x"

$$f'(x) = 10x - 6$$

$$\text{Put } f'(x) = 0 \Rightarrow 10x - 6 = 0$$

$$\text{we get } x = \frac{3}{5}$$

$$\text{take } f'(x) = 10x - 6$$

Differentiate again w.r.t. "x"

$$f''(x) = 10$$

$$\text{at } x = \frac{3}{5}$$

$$f''(x) = 10 > 0$$

So f has relative minima at $x = \frac{3}{5}$

$$f\left(\frac{3}{5}\right) = 5\left(\frac{3}{5}\right)^2 - 6\left(\frac{3}{5}\right) + 2$$

$$= \frac{9}{5} - \frac{18}{5} + 2$$

$$= \frac{9 - 18 + 10}{5}$$

$$f\left(\frac{3}{5}\right) = \frac{1}{5}$$

(iv) $f(x) = 3x^2$

Solution:

$$f(x) = 3x^2$$

Differentiate w.r.t. "x"

$$f'(x) = 6x$$

$$\text{Put } f'(x) = 0 \Rightarrow 6x = 0$$

We get $x = 0$

$$\text{take } f'(x) = 6x$$

Differentiate again w.r.t. "x"

$$f''(x) = 6$$

at $x = 0$

$$f''(x) = 6 > 0$$

So $f(x)$ has minimum value

$$\text{and } f(0) = 0$$

(v) $f(x) = 2x^3 - 2x^2 - 36x + 3$

Solution:

$$f(x) = 2x^3 - 2x^2 - 36x + 3$$

Differentiate w.r.t. "x"

$$f'(x) = 6x^2 - 4x - 36$$

$$\text{Put } f'(x) = 0$$

$$6x^2 - 4x - 36 = 0$$

$$\Rightarrow x = \frac{1 \pm \sqrt{55}}{3}$$

$$\text{Take } f'(x) = 6x^2 - 4x - 36$$

Differentiate again w.r.t. "x"

$$\text{Now } f''(x) = 12x - 4$$

$$\text{At } x = \frac{1 + \sqrt{55}}{3}$$

$$f''\left(\frac{1 + \sqrt{55}}{3}\right) = 12\left(\frac{1 + \sqrt{55}}{3}\right) - 4$$

$$= 4 + 4\sqrt{55} - 4 = 4\sqrt{55} > 0$$

Thus f has relative minima

$$f\left(\frac{1 + \sqrt{55}}{3}\right) = 2\left(\frac{1 + \sqrt{55}}{3}\right)^3 - 2\left(\frac{1 + \sqrt{55}}{3}\right)^2 - 36\left(\frac{1 + \sqrt{55}}{3}\right) + 3$$

$$f\left(\frac{1 + \sqrt{55}}{3}\right) = \frac{-1}{27}(247 + 220\sqrt{55})$$

$$\text{at } x = \frac{1 - \sqrt{55}}{3}$$

$$f''\left(\frac{1 - \sqrt{55}}{3}\right) = -4\sqrt{55} < 0$$

So f has relative maxima

$$f\left(\frac{1-\sqrt{55}}{3}\right) = 2\left(\frac{1-\sqrt{55}}{3}\right)^3 - 2\left(\frac{1-\sqrt{55}}{3}\right)^2 - 36\left(\frac{1-\sqrt{55}}{3}\right) + 3$$

$$f\left(\frac{1-\sqrt{55}}{3}\right) = \frac{-1}{27}(247 - 220\sqrt{55})$$

(vi) $f(x) = x^4 - 4x^2$

Solution:

$$f(x) = x^4 - 4x^2$$

Differentiate w.r.t. "x"

$$f'(x) = 4x^3 - 8x$$

Put $f'(x) = 0$

$$4x^3 - 8x = 0$$

$$x = 0, \quad x = \sqrt{2}, \quad x = -\sqrt{2}$$

Take $f'(x) = 4x^3 - 8x$

Differentiate again w.r.t. "x"

$$f''(x) = 12x^2 - 8$$

at $x = -\sqrt{2}$

$$f''(-\sqrt{2}) = 12(-\sqrt{2})^2 - 8$$

$$f''(-\sqrt{2}) = 24 - 8$$

$$f''(-\sqrt{2}) = 16 > 0$$

So f has relative minima

at $x = 0, \quad f''(0) = -8 < 0$

so f has relative maxima at $x = 0$

at $x = \sqrt{2}, \quad f''(\sqrt{2}) = 11 > 0$

so f has relative minima at $x = \sqrt{2}$

and $f(\sqrt{2}) = f(-\sqrt{2}) = -4$

also $f(0) = 0$

(vii) $f(x) = (x-2)^2(x-1)$

Solution:

$$f(x) = (x-2)^2(x-1)$$

$$f(x) = (x^2 - 4x + 4)(x-1)$$

$$f(x) = x^3 - 5x^2 + 8x - 4$$

Differentiate w.r.t. "x"

$$f'(x) = 3x^2 - 10x + 8$$

Put $f'(x) = 0$

$$3x^2 - 10x + 8 = 0$$

$$\Rightarrow x = 2, \quad x = \frac{4}{3}$$

Take $f'(x) = 3x^2 - 10x + 8$

Differentiate again w.r.t. "x"

$$f''(x) = 6x - 10$$

at $x = 2,$

$$f''(2) = 2 > 0$$

So f has relative minima

and $f(2) = 0$

at $x = \frac{4}{3}$

$$f''\left(\frac{4}{3}\right) = -2 < 0$$

So f has relative minima

And

$$f\left(\frac{4}{3}\right) = \left(\frac{4}{3}\right)^3 - 5\left(\frac{4}{3}\right)^2 + 8\left(\frac{4}{3}\right) - 4$$

$$= \frac{64}{27} - \frac{80}{9} + \frac{32}{3} - 4$$

$$f\left(\frac{4}{3}\right) = \frac{4}{27}$$

(viii) $f(x) = 5 + 3x - x^3$

Solution:

$$f(x) = 5 + 3x - x^3$$

Differentiate w.r.t. "x"

$$f'(x) = 3 - 3x^2$$

put $f'(x) = 0$

$$3 - 3x^2 = 0$$

$$\Rightarrow x = \pm 1$$

Take $f'(x) = 3 - 3x^2$

Differentiate again w.r.t. "x"

$$f''(x) = -6x$$

at $x = -1, \quad f''(-1) = 6 > 0$

So f has relative minima

And $f(-1) = 3$

At $x=1$, $f''(1) = -6 < 0$

So f has relative maxima

And $f(1) = 7$

Q.3 Find the maximum and minimum values of the function defined by the following equation occurring in the interval $[0, 2\pi]$

$$f(x) = \sin x + \cos x$$

Solution:

$$f(x) = \sin x + \cos x$$

Differentiate w.r.t. "x"

$$f'(x) = \cos x - \sin x$$

Put $f'(x) = 0$

$$\cos x - \sin x = 0$$

$$\sin x = \cos x \Rightarrow \tan x = 1$$

$$x = \frac{\pi}{4} \text{ \& } x = \pi + \frac{\pi}{4} = \frac{5\pi}{4}$$

Take $f'(x) = \cos x - \sin x$

Differentiate again w.r.t. "x"

$$f''(x) = -\sin x - \cos x$$

At $x = \frac{\pi}{4}$

$$f''\left(\frac{\pi}{4}\right) = -\sin\frac{\pi}{4} - \cos\frac{\pi}{4}$$

$$= -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\frac{2}{\sqrt{2}}$$

$$= -\sqrt{2} < 0$$

So f has maximum value

And

$$f\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{4} + \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}}$$

$$f\left(\frac{\pi}{4}\right) = \sqrt{2}$$

At $x = \frac{5\pi}{4}$

$$f''\left(\frac{5\pi}{4}\right) = -\sin\frac{5\pi}{4} - \cos\frac{5\pi}{4}$$

$$= \sqrt{2} > 0$$

So f has minimum value

and $f\left(\frac{5\pi}{4}\right) = \sin\frac{5\pi}{4} + \cos\frac{5\pi}{4}$

$$f\left(\frac{5\pi}{4}\right) = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}$$

$$f\left(\frac{5\pi}{4}\right) = \frac{-1-1}{\sqrt{2}} = \frac{-2}{\sqrt{2}}$$

$$f\left(\frac{5\pi}{4}\right) = -\sqrt{2}$$

So maximum and minimum values

of $f(x)$ are $\sqrt{2}$ and $-\sqrt{2}$

respectively.

Q.4 Show that $y = \frac{\ln x}{x}$ has maximum value at $x = e$

Solution:

$$y = \frac{\ln x}{x}$$

Differentiate w.r.t "x"

$$\frac{dy}{dx} = \frac{x\left(\frac{1}{x}\right) - \ln x(1)}{x^2}$$

$$\frac{dy}{dx} = \frac{1 - \ln x}{x^2}$$

Put $\frac{dy}{dx} = 0$

$$\frac{1 - \ln x}{x^2} = 0$$

$$1 - \ln x = 0$$

$$\ln x = 1 \Rightarrow \ln x = \ln e$$

$$\Rightarrow x = e$$

Take $\frac{dy}{dx} = \frac{1 - \ln x}{x^2}$

Differentiate again w.r.t. "x"

$$\frac{d^2y}{dx^2} = \frac{x^2\left(-\frac{1}{x}\right) - (1 - \ln x)(2x)}{x^4}$$

$$\frac{d^2y}{dx^2} = \frac{-x - 2x(1 - \ln x)}{x^4}$$

$$\frac{d^2y}{dx^2} = \frac{-3x + 2x \ln x}{x^4}$$

$$\frac{d^2y}{dx^2} = \frac{-3 + 2 \ln x}{x^3}$$

at $x = e$

$$\left. \frac{d^2y}{dx^2} \right|_{x=e} = \frac{-3 + 2 \ln e}{e^3}$$

$$= \frac{-1}{e^3} < 0$$

$$\left. \frac{d^2y}{dx^2} \right|_{x=e} < 0$$

so f has maximum value at $x = e$.

Q.5 Show that $y = x^x$ has minimum

value at $x = \frac{1}{e}$

Solution:

$$y = x^x$$

Taking natural log on both sides

$$\ln y = \ln x^x$$

$$\ln y = x \ln x$$

Differentiate w.r.t. "x"

$$\frac{1}{y} \frac{dy}{dx} = x \cdot \frac{1}{x} + \ln x(1)$$

$$\frac{1}{y} \frac{dy}{dx} = 1 + \ln x$$

$$\frac{dy}{dx} = y(1 + \ln x) \Rightarrow \frac{dy}{dx} = x^x(1 + \ln x)$$

Put $\frac{dy}{dx} = 0$

$$\Rightarrow x^x(1 + \ln x) = 0$$

$$\Rightarrow 1 + \ln x = 0 \quad \therefore x^x \neq 0$$

$$\ln x = -1$$

$$\ln x = -\ln e$$

$$(\because \ln e = 1)$$

$$\ln x = \ln e^{-1}$$

$$\ln x = \ln\left(\frac{1}{e}\right)$$

$$\Rightarrow x = \frac{1}{e}$$

Take $\frac{dy}{dx} = x^x(1 + \ln x)$

Differentiate again w.r.t. "x"

$$\frac{d^2y}{dx^2} = x^x\left(\frac{1}{x}\right) + (1 + \ln x)x^x(1 + \ln x)$$

$$\frac{d^2y}{dx^2} = \frac{x^x}{x} + x^x(1 + \ln x)^2$$

$$\frac{d^2y}{dx^2} = x^x\left(\frac{1}{x} + (1 + \ln x)^2\right)$$

at $x = \frac{1}{e}$

$$\left. \frac{d^2y}{dx^2} \right|_{x=\frac{1}{e}} = \left(\frac{1}{e}\right)^{\frac{1}{e}} \left[\frac{1}{\frac{1}{e}} + \left(1 + \ln \frac{1}{e}\right)^2 \right]$$

$$\left. \frac{d^2y}{dx^2} \right|_{x=\frac{1}{e}} = \left(\frac{1}{e}\right)^{\frac{1}{e}} \left[e + (1 + \ln 1 - \ln e)^2 \right]$$

$$\left. \frac{d^2y}{dx^2} \right|_{x=\frac{1}{e}} = e\left(\frac{1}{e}\right)^{\frac{1}{e}} > 0$$

$$\left. \frac{d^2y}{dx^2} \right|_{x=\frac{1}{e}} > 0$$

$\therefore y = x^x$ has minimum value at

$$x = \frac{1}{e}$$