Discriminant (BWP 2018) **(U.B + K.B)**

"For a standard quadratic equation $ax^2+bx+c=0$, the value of the expression b^2-4ac is called discriminant."

It is used to find the nature of roots without solving the equation.

Nature or Characteristics of the Roots **(U.B + K.B)**

Nature of a quadratic equation $ax^2+bx+c=0$, when $a,b,c\in Q$ and $a\neq 0$ as:

(i) If $b^2-4ac=0$, then the roots are rational (real) and equal.

(ii) If $b^2-4ac<0$, then the roots are complex conjugate or imaginary.

(iii) If $b^2-4ac>0$, and is a perfect square, then the roots are rational (real) and unequal.

(iv) If $b^2-4ac>0$, and is not a perfect square, the roots are irrational (real) and unequal.

Note **(K.B)**

If given polynomial expression is a perfect square then discriminant is 0.

Example 2: (Page # 19)

Using discriminant, find the nature of the roots of the following equation and verify the result by solving the equation.

$x^2-5x+5=0$

(LHR 2015, GRW 2016, 17, SWL 2017, RWP 2015, D.G.K 2017)

Solution:

$x^2-5x+5=0$

Here $a=1,\ b=-5,\ c=5$

Discriminant $= b^2-4ac$

$=(-5)^2-4(1)(5)$

$=25-20=5$

As discriminant > 0 but not perfect square, Roots are irrational (real) and unequal.

Verification:

Solving the equation by using quadratic formula

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$x=\frac{-(-5)\pm\sqrt{(-5)^2-4(1)(5)}}{2(1)}$$

$$x=\frac{5\pm\sqrt{5}}{2}$$

Evidently, Roots are irrational (real) and unequal.

Example 2: (Page # 21)

Find k, if the roots of the equation $(k+3)x^2-2(k+1)x-(k+1)=0$ are equal, if $k\neq -3$ **(A.B)**

Solution:

$(k+3)x^2-2(k+1)x-(k+1)=0$

Here

$a=k+3,\ b=-2(k+1),\ c=-(k+1)$

As roots are equal, discriminant is zero.

$\Rightarrow$ Disc. $=b^2-4ac=0$

$[-2(k+1)]^2-4(k+3)[-(k+1)]=0$

$4(k+1)^2+4(k+3)(k+1)=0$

$4(k+1)\left[(k+1)+(k+3)\right]=0$

$4(k+1)\left(2k+4\right)=0$

Either

$k+1=0$ or $2k+4=0$ $\because 4\neq 0$

$k=-1$ or $2k=-4$

$k=-2$

Thus, roots will be equal if $k=-1,-2$

Exercise 2.1

Q.1 Find the discriminant of the following given quadratic equations:

Solution:

(i) $2x^2+3x-1=0$ **(A.B)**

(GRW 2017, FSD 2016, MTN 2014, D.G.K 2016)

By comparing given equation with $ax^2+bx+c=0$**, we get**

$a=2,\ b=3,\ c=-1$

Disc $= b^2-4ac$

$= (3)^2-4(2)(-1)$

$= 9 + 8$

$= 17$

(ii) $6x^2-8x+3=0$ **(A.B)**

(LHR 2016, SWL 2016, D.G.K 2015, 17)

By comparing given equation with $ax^2+bx+c=0$**, we get**

$a=6,\ b=-8,\ c=3$

Disc $= b^2-4ac$

$= (-8)^2-4(6)(3)$

$= 64 - 72$

$= -8$

(iii) $9x^2-30x+25=0$ **(A.B)**

(LHR 2017, MTN 2015)

By comparing given equation with $ax^2+bx+c=0$**, we get**

$a=9,\ b=-30,\ c=25$

Disc $= (-30)^2-4(9)(25)$

$= 900 - 900$

$= 0$

(iv) $4x^2-7x-2=0$ **(A.B)**

(GRW 2014, SGD 2017, MTN 2016)

By comparing given equation with $ax^2+bx+c=0$**, we get**

$a=4,\ b=-7,\ c=-2$

Disc $= b^2-4ac$

$=(-7)^2-4(4)(-2)$

$= 49 + 32$

$= 81$

Q.2 Find the nature of the roots of the following given quadratic equations and verify the result by solving the equations:

Solution:

(i) $x^2-23x+120=0$ **(A.B)**

By comparing given equation with $ax^2+bx+c=0$**, we get**

$a=1, b=-23, c=120$

Disc $= b^2-4ac$

$= (-23)^2-4(1)(120)$

$= 529 - 480$

$= 49$

$= 7^2$

Since disc > 0 and perfect square, roots are rational (real) and unequal.

Verification:

$x^2-23x+120=0$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$=\frac{-(-23)\pm\sqrt{(-23)^2-4(1)(120)}}{2(1)}$$

$$=\frac{23\pm\sqrt{7^2}}{2}$$

$$x=\frac{23\pm7}{2}$$

Either

$x=\dfrac{23-7}{2}$ or $x=\dfrac{23+7}{2}$

$=\dfrac{16}{2}$ or $=\dfrac{30}{2}$

$x=8$ $x=15$

Hence roots are rational and unequal.

(ii) $2x^2+3x+7=0$ **(A.B)**

By comparing given equation with $ax^2+bx+c=0$**, we get**

$a=2, b=3, c=7$

Disc $= b^2-4ac$

$= (3)^2-4(2)(7)$

$= 9 - 56$

$= -47$

Since disc < 0, roots are complex and imaginary.

Verification:

$2x^2+3x+7=0$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$=\frac{-3\pm\sqrt{(3)^2-4(2)(7)}}{2(2)}$$

$$=\frac{-3\pm\sqrt{9-56}}{4}$$

$$=\frac{-3\pm\sqrt{-47}}{4}$$

$$=\frac{-3\pm\sqrt{47}i}{4}$$

Hence roots are complex/imaginary and unequal.

(iii) $16x^2-24x+9=0$ **(A.B)**

By comparing given equation with $ax^2+bx+c=0$**, we get**

$a=16,\ b=-24,\ c=9$

Disc $=b^2-4ac$

$=(-24)^2-4(16)(9)$

$=576-576$

$=0$

Since disc = 0, roots are rational (real) and equal.

Verification:

$16x^2-24x+9=0$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$=\frac{-(-24)\pm\sqrt{(-24)^2-4(16)(9)}}{2(16)}$$

$$x=\frac{24\pm\sqrt{0}}{32}$$

$$x=\frac{24\pm0}{32}$$

Either

$$x=\frac{24+0}{32} \quad \text{or} \quad x=\frac{24-0}{32}$$

$$x=\frac{24}{32} \quad\quad x=\frac{24}{32}$$

Hence roots are rational and equal.

(iv) $3x^2+7x-13=0$ **(A.B)**

By comparing given equation with $ax^2+bx+c=0$**, we get**

$a=3, b=7, c=-13$

Disc $=b^2-4ac$

$=(7)^2-4(3)(-13)$

$= 49 + 156$

$= 205$

Since disc > 0, but not a perfect sq. roots are irrational and unequal.

Verification:

$3x^2+7x-13=0$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$=\frac{-7\pm\sqrt{(7)^2-4(3)(-13)}}{2(3)}$$

$$=\frac{-7\pm\sqrt{49+56}}{6}$$

$$=\frac{-7\pm\sqrt{205}}{6}$$

Either

$$x=\frac{-7-\sqrt{205}}{6} \quad \text{or} \quad x=\frac{-7+\sqrt{205}}{6}$$

Hence roots are irrational and unequal.

Q.3 For what value of k, the expression $k^2x^2+2(k+1)x+4$ is perfect square. **(A.B + K.B)**

Solution:

$k^2x^2+2(k+1)x+4=0$

By comparing given equation with $ax^2+bx+c=0$**, we get**

$a=k^2,\quad b=2(k+1),\quad c=4$

Discriminant $=b^2-4ac$

$=\left[2(k+1)\right]^2-4(k^2)(4)$

$=4(k+1)^2-16k^2$

$=4(k^2+2k+1)-16k^2$

$=4k^2+8k+4-16k^2$

$=4+8k-12k^2$

$=4(1+2k-3k^2)$

As expression is a perfect square, the discriminant = 0

$\Rightarrow 4\left(1+2k-3k^2\right)=0$

$1+2k-3k^2=0 \quad \because\ 4\neq 0$

$1+3k-k-3k^2=0$

$1(1+3k)-k(1+3k)=0$

$(1+3k)(1-k)=0$

Either

$1+3k=0$ or $1-k=0$

$3k=-1$ $\quad$ $k=1$

$k=-\frac{1}{3}$

Result

$k=1,\ -\frac{1}{3}$

Q.4 Find the value of *k*, if the roots of the following equations are equal. **(A.B + K.B)**

Solution:

(i) $(2k-1)x^2+3kx+3=0$

Here $a=2k-1, b=3k, c=3$

Disc $= b^2-4ac$

$=(3k)^2-4(2k-1)(3)$

$=9k^2-24k+12$

Since roots are equal,

Disc = 0

$\Rightarrow 9k^2-24k+12=0$

$3(3k^2-8k+4)=0$

$3k^2-8k+4=0 \quad \because\ 3\neq 0$

$3k^2-6k-2k+4=0$

$3k(k-2)-2(k-2)=0$

$(k-2)(3k-2)=0$

Either

$k-2=0$ or $3k-2=0$

$\Rightarrow k=2$ or $3k=2$

$k=\frac{2}{3}$

Result:

$k=2,\ \frac{2}{3}$

(ii) $x^2+2(k+2)x+(3k+4)=0$

Here $a=1, b=2(k+2), c=3k+4$

Disc $= b^2-4ac$

$=\left[2(k+2)\right]^2-4(1)(3k+4)$

$=4(k+2)^2-4(3k+4)$

$=4(k^2+4k+4)-4(3k+4)$

$=4\left[k^2+4k+4-3k-4\right]$

$=4\left[k^2+k\right]$

$=4k(k+1)$

Since roots are equal, disc = 0

$4k(k+1)=0$

Either

$4k=0$ *or* $k+1=0$

$k=0$ *or* $k=-1$

Result:

$k=0,-1$

(iii) $(3k+2)x^2-5(k+1)x+(2k+3)=0$ **(A.B + K.B)**

Here

$a=3k+2, b=-5(k+1), c=2k+3$

Disc $= b^2-4ac$

$=\left[-5(k+1)\right]^2 - 4(3k+2)(2k+3)$

$=25(k+1)^2-4\left(6k^2+9k+4k+6\right)$

$=25\left(k^2+2k+1\right)-4\left(6k^2+13k+6\right)$

$=25k^2+50k+25-24k^2-52k-24$

$=k^2-2k+1$

Since roots are equal, disc = 0

$\Rightarrow k^2-2k+1=0$

$(k-1)^2=0$

Taking square root on both sides

$k-1=0$

$k=1$

Result:

$k=1$

Q.5 Show that the equation $x^2+(mx+c)^2=a^2$ **has equal roots, if** $c^2=a^2(1+m^2)$ **(A.B + K.B)**

Proof **(FSD 2016)**

$x^2+(mx+c)^2=a^2$

$x^2+m^2x^2+2mcx+c^2=a^2$

$(1+m^2)x^2+2mcx+c^2-a^2=0$

Here

$A=1+m^2, B=2mc, C=c^2-a^2$

Disc= B^2-4AC

$=(2mc)^2-4(1+m^2)(c^2-a^2)$

$=4m^2c^2-4(c^2-a^2+m^2c^2-a^2m^2)$

$=\cancel{4m^2c^2}-4c^2+4a^2-\cancel{4m^2c^2}+4a^2m^2$

$=-4c^2+4a^2+4a^2m^2$

Disc $=-4\left[a^2(1+m^2)\right]+4a^2+4m^2$

$(\because c^2=a^2(1+m^2))$

$=-4\left[a^2+a^2m^2\right]+4a^2+4a^2m^2$

$=-4a^2-4a^2m^2+4a^2-4a^2m^2$

$=0$

Disc=0

$\therefore$ Roots are equal

Hence roots are equal, if $c^2=a^2(1+m^2)$

Proved

Q.6 Find the condition that the roots of the equation $(mx+c)^2-4ax=0$ **are equal. (A.B + K.B + U.B)**

Solution:

$(mx+c)^2-4ax=0$

$m^2x^2+2mcx-4ax+c^2=0$

$m^2x^2+(2mc-4a)x+c^2=0$

Here $A=m^2, B=2mc-4a, C=c^2$

Disc = B^2-4AC

$=(2mc-4a)^2-4(m^2)(c^2)$

$=4m^2c^2-16amc+16a^2-4m^2c^2$

$=-16amc+16a^2$

$=-16a(mc-a)$

Since roots are equal, disc = 0

$\Rightarrow -16a(mc-a)=0$

Either

$mc-a=0$ or $-16a=0$

$a=mc$ $\quad a=0$

Roots of the given equation are equal

If $a=mc$ or $a=0$

Q.7 If the roots of the equation. $(c^2-ab)x^2-2(a^2-bc)x+(b^2-ac)=0$ **are equal, then** $a=0$ **or** $a^3+b^3+c^3=3abc$**.**

(A.B + K.B + U.B)

Proof

$(c^2-ab)x^2-2(a^2-bc)x+(b^2-ac)=0$

Here

$A=c^2-ab, B=-2(a^2-bc), C=b^2-ac$

Disc = B^2-4AC

$=\left[-2(a^2-bc)\right]^2-4(c^2-ab)(b^2-ac)$

$=4(a^2-bc)^2-4(b^2c^2-ac^3-ab^3+a^2bc)$

$=4(a^4-2a^2bc+b^2c^2)-4(b^2c^2-ac^3-ab^3+a^2bc)$

$=4(a^4-2a^2bc+b^2c^2-b^2c^2+ac^3+ab^3-a^2bc)$

$=4(a^4+ab^3+ac^3-3a^2bc)$

$=4a(a^3+b^3+c^3-3abc)$

Since roots are equal, disc = 0

$\Rightarrow 4a(a^3+b^3+c^3-abc)=0$

Either

$4a=0$ *or* $a^3+b^3+c^3-3abc=0$

$a=0$ $\quad a^3+b^3+c^3=3abc$

Roots are equal if either

$a=0$ *or* $a^3+b^3+c^3=3abc$

Q.8 Show that the roots of the following equations are rational.

Proof. (A.B + K.B + U.B)

(i) $a(b-c)x^2+b(c-a)x+c(a-b)=0$

Here

$A=a(b-c),\ B=b(c-a),\ C=c(a-b)$

Discriminant = B^2-4AC

$=\left[b(c-a)\right]^2-4\left[a(b-c)\right]\left[c(a-b)\right]$

$=b^2(c-a)^2-4ac(b-c)(a-b)$

$=b^2\left(c^2-2ac+a^2\right)-4ac\left(ab-b^2-ac+bc\right)$

$=b^2c^2-2ab^2c+a^2b^2-4a^2bc+4ab^2c+4a^2c^2-4abc^2$

$=b^2c^2+a^2b^2+4a^2c^2+2ab^2c-4a^2bc-4abc^2$

$=(bc)^2+(ab)^2+(-2ac)^2+2(bc)(ab)$

$+2(ab)(-2ac)+2(-2ac)(bc)$

$\therefore\ a^2+b^2+c^2+2ab+2bc+2ca=(a+b+c)^2$

$=\ (bc+ab-2ac)^2$

As discriminate > 0 and perfect square, roots are rational.

(ii) $(a+2b)x^2+2(a+b+c)x+(a+2c)=0$

Here

$A=a+2b, B=2(a+b+c)\ , C=a+2c$

Disc= B^2-4AC

$=[2(a+b+c)]^2-4(a+2b)(a+2c)$

$=4(a+b+c)^2-4\left(a^2+2ac+2ab+4bc\right)$

$=4\left(a^2+b^2+c^2+2ab+2bc+2ca\right)$

$-4\left(a^2+2ac+2ab+4bc\right)$

$=4(a^2+b^2+c^2+2ab+2bc+2ca$

$-a^2-2ac-2ab-4bc)$

$=4\left(b^2+c^2-2bc\right)$

$=4(b-c)^2$

$=\left[2(b-c)\right]^2>0$

Since disc is perfect square roots are rational.

Q.9 For all values of k, prove that the roots of the equation.

$x^2-2\left(k+\frac{1}{k}\right)x+3=0, k\neq 0$ **are real.**

Proof: **(A.B + U.B)**

Here

$a=1,\ \ b=-2\left(k+\frac{1}{k}\right),\ \ c=3$

Disciriminat = b^2-4ac

$=\left[-2\left(k+\frac{1}{k}\right)\right]^2-4(1)(3)$

$=4\left(k+\frac{1}{k}\right)^2-12$

$=4\left(k^2+\frac{1}{k^2}+2\right)-12$

$=4\left[\left(k^2+\frac{1}{k^2}+2\right)-3\right]$

$=4\left[k^2+\frac{1}{k^2}-1\right]$

$=4\left[k^2+\frac{1}{k^2}-2+1\right]$

$=4\left[\left(k-\frac{1}{k}\right)^2+1\right]$

>0

As disc. > 0, roots are real.

Q.10 Show that the roots of the equation $(b-c)x^2+(c-a)x+(a-b)=0$ **are real.** **(A.B + U.B + K.B)**

Proof:

$(b-c)x^2+(c-a)x+(a-b)=0$

Here $A=b-c, B=c-a, C=a-b$

Disc = B^2-4AC

$=(c-a)^2-4(b-c)(a-b)$

$=c^2-2ac+a^2-4\left(ab-b^2-ac+bc\right)$

$=c^2-2ac+a^2-4ab+4b^2+4ac-4bc$

$=c^2+a^2+4b^2+2ac-4ab-4bc$

$=(c)^2+(a)^2+(2b)^2+2(c)(a)+2(a)(-2b)+2(-2b)(c)$

$\because a^2+b^2+c^2+2ab+2bc+2ca=(a+b+c)^2$

$=(c+a-2b)^2>0$

Hence roots are real.