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Mathematics-10
Exercise 2.2
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Derivation of Cube Roots of Unity

(A.B + K.B)

Let $x=\sqrt[3]{1}$

Taking cube on both sides

$x^3=1$

$x^3-1=0$

$(x)^3-(1)^3=0$

$(x-1)(x^2+x+1)=0$

$(x-1)(x^2+x+1)=0$

Either

$x-1=0$ or $x^2+x+1=0$

$x=1$

Here $a=1, b=1, c=1$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$x=\frac{-1\pm\sqrt{(1)^2-4(1)(1)}}{2(1)}$$

$$=\frac{-1\pm\sqrt{1-4}}{2}$$

$$=\frac{-1\pm\sqrt{-3}}{2}=\frac{-1\pm\sqrt{3}i}{2} \quad \because \sqrt{-1}=i$$

Either

$$x=\frac{-1+\sqrt{3}i}{2}, \qquad x=\frac{-1-\sqrt{3}i}{2}$$

$$=\omega \qquad\qquad =\omega^2$$

$\therefore$ **Cube root of unity are** $1,\omega,\omega^2$

Note **(K.B)**

We can write anyone complex cube root as ω (Omega), then other will be ω^2.

Properties of Cube Root of Unity

(i) Proving that each complex cube root of unity is the square of other.

i.e. $\left(\frac{-1+\sqrt{-3}}{2}\right)^2=\frac{-1-\sqrt{-3}}{2}$

and $\left(\frac{-1-\sqrt{-3}}{2}\right)^2=\frac{-1+\sqrt{-3}}{2}$

(LHR 2014, GRW 2017, FSD 2016, 17, SGD 2015, 16, BWP 2017, MTN 2017)

(K.B + U.B + A.B)

Proof:

We have to prove

(i) $\left(\frac{-1+\sqrt{-3}}{2}\right)^2=\frac{-1-\sqrt{-3}}{2}$

Consider

$$\left(\frac{-1+\sqrt{-3}}{2}\right)^2$$

$$=\frac{(-1)^2+\left(\sqrt{-3}\right)^2+2(-1)\left(\sqrt{-3}\right)}{4}$$

$$=\frac{1+(-3)-2\sqrt{-3}}{4}$$

$$=\frac{1-3-2\sqrt{-3}}{4}$$

$$=\frac{-2-2\sqrt{-3}}{4}$$

$$=\frac{2\left(-1-\sqrt{-3}\right)}{4}$$

$$=\frac{-1-\sqrt{-3}}{2}$$

(ii) $\left(\frac{-1-\sqrt{-3}}{2}\right)^2=\frac{-1+\sqrt{-3}}{2}$

Now consider

$$\left(\frac{-1-\sqrt{-3}}{2}\right)^2$$
$$=\frac{(1)^2+\left(\sqrt{-3}\right)^2-2(-1)\left(-\sqrt{3}\right)}{4}$$
$$=\frac{1+(-3)+2\sqrt{-3}}{4}$$
$$=\frac{1-3+2\sqrt{-3}}{4}$$
$$=\frac{-2+2\sqrt{-3}}{4}$$
$$=\frac{2\left(-1+\sqrt{-3}\right)}{4}$$
$$=\frac{-1+\sqrt{-3}}{2}$$

Thus, each of the complex cube roots of unity is the square of the other.

(ii) Proving that product of three cube root of unity is one.

i.e. $1.\omega.\omega^2=1$ **(K.B + U.B)**

Proof:

L.H.S $=1.\omega.\omega^2$

By putting the values

$$=(1)\times\left(\frac{-1+\sqrt{-3}}{2}\right)\left(\frac{-1-\sqrt{-3}}{2}\right)$$
$$=\frac{(-1)^2-\left(\sqrt{-3}\right)^2}{4}$$
$$=\frac{1-(-3)}{4}$$
$$=\frac{1+3}{4}$$
$$=\frac{4}{4}$$
$= 1$

$=$ R.H.S

Proved

(iii) Proving that sum of three cube roots of unit is zero.

i.e. $1+\omega+\omega^2=0$

(LHR 2016, GRW 2017, FSD 2015, 17, SGD 2015, 16, BWP 2016, RWP 2015)

Proof:

L.H.S $=1+\omega+\omega^2$

(By putting values)

$$=1+\frac{-1+\sqrt{-3}}{2}+\frac{-1-\sqrt{-3}}{2}$$
$$=\frac{2+(-1)+\sqrt{-3}-1-\sqrt{-3}}{2}$$
$$=\frac{2-1+\sqrt{-3}-1-\sqrt{-3}}{2}$$
$$=\frac{0}{2}$$
$= 0$

$=$ R.H.S

Proved

Important Results **(K.B + U.B)**

(i) $1+\omega+\omega^2=0$

$\Rightarrow 1+\omega=-\omega^2$

$1+\omega^2=-\omega$

$\omega+\omega^2=-1$

(ii) $1.\omega.\omega^2=1$

$\Rightarrow \omega^3=1$

(iii) $1.\omega.\omega^2=1$

$\omega.\omega^2=1$

$\Rightarrow \omega=\frac{1}{\omega^2}$ or $\omega^2=\frac{1}{\omega}$

Note: Complex cube roots are reciprocal of each other.

(iv) $\omega^4=\omega^3.\omega=1.\omega=\omega$

$\omega^5=\omega^3.\omega^2=1.\omega^2=\omega^2$

$\omega^6=\left(\omega^3\right)^2=(1)^2=1$ **and so on**

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Example 1: (Page # 25)

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Evaluate: $\left(-1+\sqrt{-3}\right)^8+\left(-1-\sqrt{-3}\right)^8$ **(A.B)**

Solution:

$$\left(-1+\sqrt{-3}\right)^8+\left(-1-\sqrt{-3}\right)^8$$

$$=\left[2\left(\frac{-1+\sqrt{-3}}{2}\right)\right]^8+\left[2\left(\frac{-1-\sqrt{-3}}{2}\right)\right]^8$$

$$=(2\omega)^8+\left(2\omega^2\right)^8$$

$$=2^8\omega^8+2^8\omega^{16}$$

$$=2^8\left(\omega^6.\omega^2+\omega^{15}\omega\right)$$

$$=256\left(\left(\omega^3\right)^2.\omega^2+\left(\omega^3\right)^5\omega\right)$$

$$=256\left((1)^2.\omega^2+(1)^5\omega\right)\ \because\omega^3=1$$

$$256\left(\omega^2+\omega\right)$$

$$=256(-1)\qquad\because 1+\omega+\omega^2=0$$

$$=-256$$

Example 2: (Page # 25) **(A.B)**

To Prove

$$x^3-y^3=(x-y)(x-\omega y)(x-\omega^2 y)$$

Proof:

$$\text{R.H.S}=(x-y)(x-\omega y)(x-\omega^2 y)$$

$$=(x-y)(x^2-\omega xy-\omega^2 xy+\omega^3 y^2)$$

$$=(x-y)(x^2-(\omega+\omega^2)xy+y^2)$$

$$=(x-y)(x^2-(-1)xy+y^2)\ \because 1+\omega+\omega^2=0$$

$$=(x-y)(x^2+xy+y^2)$$

$$=x^3-y^3=\text{L.H.S}$$

Exercise 2.2

Q.1 Find the cube roots of $-1, 8, -27, 64$.

(LHR 2015) **(K.B + A.B)**

Solution:

(i) Finding cube roots of -1

Let $x = \sqrt[3]{-1}$

Taking cube on both sides

$x^3 = -1$

$x^3 + 1 = 0$

$\because a^3 + b^3 = (a+b)(a^2 - ab + b^2)$

$(x+1)(x^2 - x + 1) = 0$

Either

$x + 1 = 0 \rightarrow (i)$ or $x^2 - x + 1 = 0 \rightarrow (ii)$

Equation $(i) \Rightarrow$

$x = -1$

Equation $(ii) \Rightarrow$

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(1)}}{2(1)}$$

$$= \frac{1 \pm \sqrt{1-4}}{2}$$

$$= \frac{1 \pm \sqrt{-3}}{2}$$

Either

$x = \frac{1 - \sqrt{-3}}{2}$ or $x = \frac{1 + \sqrt{-3}}{2}$

$x = -\left(\frac{-1 + \sqrt{-3}}{2}\right)$ $\quad x = -\left(\frac{-1 - \sqrt{-3}}{2}\right)$

$= -\omega$ $\quad = -\omega^2$

$\therefore$ **Cube roots of -1 are** $-1, -\omega, -\omega^2$

(ii) Finding cube roots of 8

(K.B + A.B)

Let $x = \sqrt[3]{8}$

Taking cube on both sides

$x^3 = 8$

$x^3 - 8 = 0$

$x^3 - 2^3 = 0$

$\because a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

$(x-2)(x^2 + 2x + 4) = 0$

Either

$x - 2 = 0 \rightarrow (i)$ or $x^2 + 2x + 4 = 0 \rightarrow (ii)$

Equation $(i) \Rightarrow$

$x = 2$

Equation $(ii) \Rightarrow$

$$x = \frac{-2 \pm \sqrt{(2)^2 - 4(1)(4)}}{2(1)}$$

$$= \frac{-2 \pm \sqrt{4-16}}{2}$$

$$= \frac{-2 \pm \sqrt{-12}}{2}$$

$$= \frac{-2 \pm 2\sqrt{-3}}{2}$$

Either

$x = 2\left(\frac{-1 + \sqrt{-3}}{2}\right)$ or $x = 2\left(\frac{-1 - \sqrt{-3}}{2}\right)$

$= 2\omega$ $\quad = 2\omega^2$

$\therefore$ **Cube roots of 8 are 2,** $2\omega, 2\omega^2$

(iii) Finding cube roots of –27

(SGD 2014) **(K.B + A.B)**

Let

$x = \sqrt[3]{-27}$

$x^3 + 27 = 0$

$\because a^3 + b^3 = (a+b)(a^2 - ab + b^2)$

$x^3 + 3^3 = 0$

$(x+3)(x^2 - 3x + 9) = 0$

Either

$x + 3 = 0 \rightarrow (i)$ or $x^2 - 3x + 9 = 0 \rightarrow (ii)$

Equation $(i) \Rightarrow$

$x = -3$

Equation $(ii) \Rightarrow$

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(9)}}{2(1)}$$

$$= \frac{3 \pm \sqrt{9-36}}{2}$$

$$= \frac{3 \pm \sqrt{-27}}{2}$$

$$= \frac{3 \pm 3\sqrt{-3}}{2}$$

$$= -3\left(\frac{-1\pm\sqrt{-3}}{2}\right)$$

Either

$$x = -3\left(\frac{-1+\sqrt{-3}}{2}\right) \text{ or } x = -3\left(\frac{-1-\sqrt{-3}}{2}\right)$$

$$x = -3\omega \qquad x = -3\omega^2$$

$\therefore$ **Cube roots of –27 are –3, -3ω, $-3\omega^2$**

(iv) Finding cube roots of 64.

(LHR 2015) (K.B + A.B)

Let

$x = \sqrt[3]{64}$

Taking cube on both sides

$x^3 = 64$

$x^3 - 64 = 0$

$x^3 - 4^3 = 0$

$\because a^3 - b^3 = (a-b)(a^2+ab+b^2)$

$(x-4)(x^2+4x+16) = 0$

Either

$x - 4 = 0 \rightarrow (i)$ or $x^2+4x+16=0 \rightarrow (ii)$

Equation $(i) \Rightarrow$

$x = 4$

Equation $(ii) \Rightarrow$

$$x = \frac{-4\pm\sqrt{(4)^2-4(1)(16)}}{2(1)}$$

$$x = \frac{-4\pm\sqrt{16-64}}{2}$$

$$x = \frac{-4\pm\sqrt{-48}}{2}$$

$$x = \frac{-4\pm 4\sqrt{-3}}{2}$$

$$x = 4\left(\frac{-1\pm\sqrt{-3}}{2}\right)$$

Either

$$x = 4\left(\frac{-1+\sqrt{-3}}{2}\right) \text{ or } x = 4\left(\frac{-1-\sqrt{-3}}{2}\right)$$

$$x = 4\omega \quad \text{or} \quad x = 4\omega^2$$

$\therefore$ **Cube roots of 64 are** $4, 4\omega, 4\omega^2$

Q.2 Evaluate: (K.B + A.B)

(FSD 2017, BWP 2016, RWP 2015, MTN 2014, 15, 17, D.G.K 2015, 17)

(i) $(1-\omega-\omega^2)^7$

(ii) $(1-3\omega-3\omega^2)^5$

(iii) $(9+4\omega+4\omega^2)^3$

(iv) $(2+2\omega-2\omega^2)(3-3\omega+3\omega^2)$

(v) $(-1+\sqrt{-3})^6 + (-1-\sqrt{-3})^6$

(vi) $\left(\frac{1+\sqrt{-3}}{2}\right)^9 + \left(\frac{1-\sqrt{-3}}{2}\right)^9$

(vii) $\omega^{37}+\omega^{38}-5$

(viii) $\omega^{-13}+\omega^{-17}$

Solution:

(i) $(1-\omega-\omega^2)^7$

(GRW 2014, 16, 17, FSD 2016, BWP 2017, MTN 2017)

$= [1-(\omega+\omega^2)]$

$= [1-(-1)]^7 \qquad \because \omega+\omega^2 = -1$

$= (1+1)^7$

$= 2^7$

$= 128$

(ii) $(1-3\omega-3\omega^2)^5$

$= [1-3(\omega+\omega^2)]^5$

$= [1-3(-1)]^5 \qquad \because \omega+\omega^2 = -1$

$= (1+3)^5$

$= (4)^5$

$= 1024$

(iii) $(9+4\omega+4\omega^2)^3$

(GRW 2014, RWP 2017, FSD 2017, BWP 2016)

$= [9+4(\omega+\omega^2)]^3$

$= [9+4(-1)]^3 \qquad \because \omega+\omega^2 = -1$

$= (9-4)^3$

$= (5)^3$

$= 125$

(iv) $(2+2\omega-2\omega^2)(3-3\omega+3\omega^2)$

(SWL 2017)

$=2\left(1+\omega-\omega^2\right)3\left(1-\omega+\omega^2\right)$

$=6\left[\left(1+\omega\right)-\omega^2\right]\left[\left(1+\omega^2\right)-\omega\right]$

$\because 1+\omega+\omega^2=0$

$=6\left(-\omega^2-\omega^2\right)\left(-\omega-\omega\right)$

$=6\left(-2\omega^2\right)\left(-2\omega\right)$

$=6(4\omega^3)$

$=24\omega^3$

$=24(1) \quad \because \omega^3=1$

$=24$

(v) $\left(-1+\sqrt{-3}\right)^6+\left(-1-\sqrt{-3}\right)^6$

$\because \omega=\dfrac{-1+\sqrt{-3}}{2}, \omega^2=\dfrac{-1-\sqrt{-3}}{2}$

$=\left(2\omega\right)^6+\left(2\omega^2\right)^6$

$=2^6\omega^6+2^6\omega^{12}$

$=2^6\left(\omega^6+\omega^{12}\right)$

$=64\left[\left(\omega^3\right)^2+\left(\omega^3\right)^4\right]$

$=64\left[\left(1\right)^2+\left(1\right)^4\right] \quad \because \omega^3=1$

$=64\left(1+1\right)$

$=64\left(2\right)$

$=128$

(vi) $\left(\dfrac{1+\sqrt{-3}}{2}\right)^9+\left(\dfrac{1-\sqrt{-3}}{2}\right)^9$

$\omega=\dfrac{-1+\sqrt{3}}{2}, \omega^2=\dfrac{-1-\sqrt{-3}}{2}$

$=\left(\omega\right)^9+\left(\omega^2\right)^9$

$=\left(\omega\right)^9+\left(\omega\right)^{18}$

$=\left(\omega^3\right)^3+\left(\omega^3\right)^6$

$=\left(1\right)^3+\left(1\right)^6$

$=1+1$

$=2$

(vii) $\omega^{37}+\omega^{38}-5$

(LHR 2015, SWL 2016, MTN 2015, D.G.K 2016, 17)

$=\omega.\omega^{36}+\omega^2.\omega^{36}-5$

$=\left(\omega+\omega^2\right).\omega^{36}-5$

$=\left(-1\right)\left[\left(\omega^3\right)^{12}\right]-5$

$=-1\left[\left(1\right)^{12}\right]-5 \quad \because \omega^3=1$

$=-1\left(1\right)-5$

$=-1-5$

$=-6$

(viii) $\omega^{-13}+\omega^{-17}$

$=\dfrac{1}{\omega^{13}}+\dfrac{1}{\omega^{17}}$

$=\dfrac{1}{\omega.\omega^{12}}+\dfrac{1}{\omega^2.\omega^{15}}$

$=\dfrac{1}{\omega.\left(\omega^3\right)^4}+\dfrac{1}{\omega^2.\left(\omega^3\right)^5}$

$=\dfrac{1}{\omega\left(1\right)^4}+\dfrac{1}{\omega^2\left(1\right)^5} \quad \because \omega^3=1$

$=\dfrac{1}{\omega\left(1\right)}+\dfrac{1}{\omega^2\left(1\right)}$

$=\dfrac{1}{\omega}+\dfrac{1}{\omega^2}$

$=\omega^2+\omega \quad \because \omega.\omega^2=1$

$=-1 \quad \because 1+\omega+\omega^2=0$

Q.3 Prove that **(K.B + U.B)**

$x^3+y^3=\left(x+y\right)\left(x+\omega y\right)\left(x+\omega^2 y\right)$

(SGD 2015, BWP 2016)

Proof:

R.H.S

$=\left(x+\text{y}\right)\left(x+\omega\text{y}\right)\left(x+\omega^2\text{y}\right)$

$=\left(x+y\right)\left(x^2+\omega^2x\text{y}+\omega x\text{y}+\omega^3\text{y}^2\right)$

$=\left(x+\text{y}\right)\left[x^2+\left(\omega^2+\omega\right)x\text{y}+\left(1\right)\text{y}^2\right] \because \omega^3=1$

$=\left(x+\text{y}\right)\left[x^2+\left(-1\right)x\text{y}+\text{y}^2\right] \because 1+\omega+\omega^2=0$

$=\left(x+\text{y}\right)\left[x^2-x\text{y}+\text{y}^2\right]$

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$= x^3 + y^3$

= L.H.S

Proved

Q.4 Prove that $x^3 + y^3 + z^3 - 3xyz$

$= (x+y+z)(x+\omega y+\omega^2 z)(x+\omega^2 y+\omega z)$

(K.B + U.B)

Proof:

R.H.S

$= (x+y+z)(x+\omega y+\omega^2 z)(x+\omega^2 y+\omega z)$

$= (x+y+z)(x^2+\omega^2 xy+\omega xz+\omega xy+\omega^3 y^2$
$+\omega^2 yz+\omega^2 xz+\omega^4 yz+\omega^3 z^2)$

$= (x+y+z)(x^2+\omega^3 y^2+\omega^3 z^2+\omega^2 xy+\omega xy$
$+\omega^2 yz+\omega^4 yz+\omega xz+\omega^2 xz)$

$= (x+y+z)(x^2+(1)y^2+(1)z^2+(\omega^2+\omega)xy$
$+(\omega^2+\omega^4)yz+(\omega+\omega^2)xz) \quad \because \omega^3 = 1$

$= (x+y+z)\left[x^2+y^2+z^2+(-1)xy+(\omega^2+\omega)yz+(-1)xz\right]$

$\because 1+\omega+\omega^2 = 0, \omega^4 = \omega.\omega^3 = \omega$

$= (x+y+z)(x^2+y^2+z^2-xy+(-1)yz-xz)$

$= (x+y+z)(x^2+y^2+z^2-xy-yz-xz)$

$= x^3+y^3+z^3-3xyz$

= L.H.S

Proved

Q.5 Prove that

$(1+\omega)(1+\omega^2)(1+\omega^4)(1+\omega^8)....2n$

(K.B + A.B + U.B)

Proof:

L.H.S

$= (1+\omega)(1+\omega^2)(1+\omega^4)(1+\omega^8)...2n$ factors

$\because \omega^4 = \omega\times\omega^3 = \omega, \omega^8 = \omega^2\times\omega^6 = \omega^2\times(\omega^3)^2 = \omega^2(1) = \omega^2$

$= (1+\omega)(1+\omega^2)(1+\omega)(1+\omega^2)...2n$ factors

$= [(1+\omega)(1+\omega)(1+\omega).....n$ factors$]$

$[(1+\omega^2)(1+\omega^2)(1+\omega^2).....n$ factors$]$

$= (1+\omega)^n(1+\omega^2)^n$

$= \left[(1+\omega)(1+\omega^2)\right]^n$

$= (1+\omega+\omega^2+\omega^3)^n$

$= (0+1)^n \quad \because 1+\omega+\omega^2 = 0, \omega^3 = 1$

$= (1)^n$

$= 1$

=R.H.S