Mathematics-10
Exercise 2.4

Symmetric Function of the Roots of Quadratic Equation (K.B + U.B)

(MTN 2014, FSD 2014)

A function in which the roots involved are such that the value of the expression remains same, when roots are interchanged is called symmetric function. i.e. $f(\alpha,\beta)=f(\beta,\alpha)$

Some symmetric functions are:

$\alpha^2+\beta^2$, $\alpha^3+\beta^3$, $\frac{1}{\alpha}+\frac{1}{\beta}$

Example: (Page # 30) **(K.B + U.B)**

Verify that

$\alpha^2+\beta^2+2\alpha\beta$ **is Symmetric**

Verification:

Let $f(\alpha,\beta)=\alpha^2+\beta^2+2\alpha\beta \rightarrow (\text{i})$

$$f(\beta,\alpha)=(\beta)^2+(\alpha)^2+2\beta\alpha$$
$$=\beta^2+\alpha^2+2\alpha\beta$$
$$=\alpha^2+\beta^2+2\alpha\beta \rightarrow (\text{ii})$$

From equation (i) and (ii) we get

$f(\alpha,\beta)=f(\beta,\alpha)$

Hence $\alpha^2+\beta^2+2\alpha\beta$ **is symmetric**

Example: (Page # 30) **(A.B + U.B)**

Find the value of $\alpha^3+\beta^3+3\alpha\beta$**, if** $\alpha=2,\ \beta=1$**. Also find the value of** $\alpha^3+\beta^3+3\alpha\beta$ **if** $\alpha=1,\ \beta=2$**.**

Solution:

When $\alpha=2,\ \beta=1$

$$\alpha^3+\beta^3+3\alpha\beta=(2)^3+(1)^3+3(2)(1)$$
$$=8+1+6=15$$

When $\alpha=1,\ \beta=2$

$$\alpha^3+\beta^3+3\alpha\beta=(1)^3+(2)^3+3(1)(2)$$
$$=1+8+6=15$$

Note

Expression $\alpha^3+\beta^3+3\alpha\beta$ represents a symmetric function.

Exercise 2.4

Q.1 If α, β are roots of the equation $x^2+px+q=0$**, then evaluate**

(i) $\alpha^2+\beta^2$

(ii) $\alpha^3\beta+\alpha\beta^3$

(iii) $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}$

(i) Solution: **(A.B + U.B)**

$x^2+px+q=0$

Here a = 1, b = p, c = q

Roots of given equation are α,β

Then

$$\alpha+\beta=-\frac{b}{a}$$
$$=-\frac{p}{1}$$
$$=-p$$
$$\alpha\beta=\frac{c}{a}$$
$$=\frac{q}{1}$$
$$=q$$

(i) Now $\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta$

$$\alpha^2+\beta^2=(-p)^2-2(q)$$
$$=p^2-2q$$

(ii) $\alpha^3\beta+\alpha\beta^3=\alpha\beta(\alpha^2+\beta^2)$ **(A.B)**

$$=\alpha\beta\left[(\alpha+\beta)^2-2\alpha\beta\right]$$
$$=q\left[(-p)^2-2(q)\right]$$
$$\Rightarrow \alpha^3\beta+\alpha\beta^3=q(p^2-2q)$$

(iii) $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}=\frac{\alpha^2+\beta^2}{\alpha\beta}$

(LHR 2015) **(K.B + U.B)**

$$= \frac{(\alpha+\beta)^2 - 2\alpha\beta}{\alpha\beta}$$

$$= \frac{(-p)^2 - 2(q)}{q}$$

$$\Rightarrow \frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{p^2 - 2q}{q}$$

Q.2 If α, β are the roots of the equation $4x^2 - 5x + 6 = 0$, then find the value of

(i) $\frac{1}{\alpha} + \frac{1}{\beta}$ **(K.B + A.B)**

(ii) $\alpha^2\beta^2$ **(A.B + U.B)**

(iii) $\frac{1}{\alpha^2\beta} + \frac{1}{\alpha\beta^2}$ **(U.B + A.B)**

(iv) $\frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha}$ **(K.B + U.B)**

(LHR 2016, GRW 2014, SWL 2016, MTN 216, SGD 2015, D.G.K 2014)

Solution:

$4x^2 - 5x + 6 = 0$

Here $a = 4, b = -5, c = 6$

Since α, β be the roots of the given equation

Then

$$\alpha + \beta = -\frac{b}{a}$$

$$= -\frac{(-5)}{4}$$

$$= \frac{5}{4}$$

$$\alpha\beta = \frac{c}{a}$$

$$= \frac{6}{4}$$

$$= \frac{3}{2}$$

(i) $$\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\beta + \alpha}{\alpha\beta}$$

$$= \frac{\alpha + \beta}{\alpha\beta}$$

$$= \frac{\frac{5}{4}}{\frac{3}{2}}$$

$$= \frac{5}{4} \times \frac{2}{3}$$

$$\Rightarrow \frac{1}{\alpha} + \frac{1}{\beta} = \frac{5}{6}$$

(ii) $$\alpha^2\beta^2 = (\alpha\beta)^2$$

$$= \left(\frac{3}{2}\right)^2$$

$$\Rightarrow \alpha^2\beta^2 = \frac{9}{4}$$

(iii) $$\frac{1}{\alpha^2\beta} + \frac{1}{\alpha\beta^2} = \frac{\beta + \alpha}{\alpha^2\beta^2}$$

$$= \frac{\alpha + \beta}{(\alpha\beta)^2}$$

$$= \frac{\frac{5}{4}}{\left(\frac{3}{2}\right)^2}$$

$$= \frac{\frac{5}{4}}{\frac{9}{4}}$$

$$= \frac{5}{4} \times \frac{4}{9}$$

$$\Rightarrow \frac{1}{\alpha^2\beta} + \frac{1}{\alpha\beta^2} = \frac{5}{9}$$

(iv) $$\frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha} = \frac{\alpha^3 + \beta^3}{\alpha\beta}$$

$$= \frac{(\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)}{\alpha\beta}$$

Putting the values

$$= \frac{\left(\frac{5}{4}\right)^3 - 3\left(\frac{3}{2}\right)\left(\frac{5}{4}\right)}{\frac{3}{2}}$$

$$=\frac{\frac{125}{64}-\frac{45}{8}}{\frac{3}{2}}$$

$$=\frac{125-360}{64}\times\frac{2}{3}$$

$$=\frac{-235}{64}\times\frac{2}{3}$$

$$\Rightarrow \frac{\alpha^2}{\beta}+\frac{\beta^2}{\alpha}=-\frac{235}{96}$$

Q.3 If α, β are the roots of the equation $lx^2+mx+n=0 \quad (l\neq 0)$, then find the values of

(i) $\alpha^3\beta^2+\alpha^2\beta^3$ **(A.B + U.B)**

(ii) $\frac{1}{\alpha^2}+\frac{1}{\beta^2}$ **(K.B + U.B)**

Solution:

$lx^2+mx+n=0$

Roots of given equation are α,β

Then $\alpha+\beta=-\frac{b}{a}=-\frac{m}{l}$

$$\alpha\beta=\frac{c}{a}$$

$$=\frac{n}{l}$$

(i) Now $\alpha^3\beta^2+\alpha^2\beta^2=\alpha^2\beta^2(\alpha+\beta)$

$$=(\alpha\beta)^2(\alpha+\beta)$$

$$=\left(\frac{n}{l}\right)^2\times\left(-\frac{m}{l}\right)$$

$$=\frac{n^2}{l^2}\times\left(-\frac{m}{l}\right)$$

$$\Rightarrow \alpha^3\beta^2+\alpha^2\beta^2=-\frac{mn^2}{l^3}$$

(ii) $\frac{1}{\alpha^2}+\frac{1}{\beta^2}=\frac{\beta^2+\alpha^2}{\alpha^2\beta^2}$

(FSD 2017, SWL 2017, BWP 2014, D.G.K 2017)

$$=\frac{\alpha^2+\beta^2}{(\alpha\beta)^2}$$

$$=\frac{(\alpha+\beta)^2-2\alpha\beta}{(\alpha\beta)^2}$$

$$=\frac{\left(-\frac{m}{l}\right)^2-2\left(\frac{n}{l}\right)}{\left(\frac{n}{l}\right)^2}$$

$$=\frac{\frac{m^2}{l^2}-\frac{2n}{l}}{\frac{n^2}{l^2}}$$

$$=\frac{m^2-2ln}{l^2}\times\frac{l^2}{n^2}$$

$$\Rightarrow \frac{1}{\alpha^2}+\frac{1}{\beta^2}=\frac{m^2-2ln}{n^2}$$