Example: (Page # 79) **(A.B)**

Resolve $\frac{1}{(x-1)^2(x-2)}$ into partial fractions.

Solution:

Let, $\frac{1}{(x-1)^2(x-2)}=\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{x-2}$

Multiplying both sides by $(x-1)^2(x-2)$, we get

$1=A(x-1)(x-2)+B(x-2)+C(x-1)^2$

$A(x^2-3x+2)+B(x-2)+C(x^2-2x+1)=1\rightarrow(i)$

Since (i) is an identity and is true for all values of x

Put $x-1=0$ or $x=1$ in (i), we get

$B(1-2)=1 \quad\Rightarrow\quad -B=1$ or $B=-1$

Put $x-2=0$ or $x=2$ in (i), we get

$C(2-1)^2=1 \quad\Rightarrow\quad C=1$

Equating coefficients of x^2 on both sides of (i)

$A+C=0 \quad\Rightarrow\quad A=-C$ so $A=-1$

Hence required partial fractions are

$$\frac{-1}{x-1}-\frac{1}{(x-1)^2}+\frac{1}{(x-2)}$$

Thus,

$$\frac{1}{(x-1)^2(x-2)}=\frac{1}{x+2}-\frac{1}{(x-1)}-\frac{1}{(x-1)^2}$$

Exercise 4.2

Resolve into partial fractions.

Q.1 $\frac{x^2-3x+1}{(x-1)^2(x-2)}$ **(A.B)**

Solution:

Let $\frac{x^2-3x+1}{(x-1)^2(x-2)}=\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{x-2}\rightarrow(\text{i})$

Multiplying by $(x-1)^2(x-2)$

$x^2-3x+1=A(x-1)(x-2)+B(x-2)+C(x-1)^2\rightarrow(\text{ii})$

$x^2-3x+1=A(x^2-3x+2)+B(x-2)+C(x^2-2x+1)$

$=Ax^2-3Ax+2A+Bx-2B+Cx^2-2Cx+C$

$=Ax^2+Cx^2-3Ax+Bx-2Cx+2A+C-2B$

$\Rightarrow x^2-3x+1=(A+C)x^2+(-3A+B-2C)x+(2A-2B+C)$

By comparing coefficients of alike powers of x

$1=A+C\rightarrow$ (iii)

$-3=-3A+B-2C\rightarrow$ (iv)

$1=2A-2B+C\rightarrow$ (v)

Put $x-1=0\Rightarrow x=1$ in equation (ii)

$(1)^2-3(1)+1=A(1-1)(1-2)+B(1-2)+C(0)^2$

$1-3+1=0+B(-1)+0$

$-1=-1B$

$B=1$

Put $B=x-2=0\Rightarrow x=2$ in equation (ii)

$(2)^2-3(2)+1=A(2-1)(2-2)+B(2-2)+C(2-1)^2$

$4-6+1=A(1)(0)+B(0)+C(1)^2$

$-1=C$

Or $C=-1$

Put $C=-1$ in equation (i)

$1=A+(-1)$

$1+1=A$

Or A = 2

Now putting values in equation (i)

$$\frac{x^2-3x+1}{(x-1)^2(x-2)}=\frac{2}{x-1}+\frac{1}{(x-1)^2}+\frac{-1}{(x-2)}$$

$$=\frac{2}{x-1}+\frac{1}{(x-1)^2}-\frac{1}{(x-2)}$$

Q.2 $\frac{x^2+7x+11}{(x+2)^2(x+3)}$ **(A.B)**

(LHR 2016, SGD 2016, RWP 2015)

Solution: $\frac{x^2+7x+11}{(x+2)^2(x+3)}$

Consider

$$\frac{x^2+7x+11}{(x+2)^2(x+3)}=\frac{A}{x+2}+\frac{B}{(x+2)^2}+\frac{C}{x+3}\rightarrow(\text{i})$$

Multiplying by $(x+2)^2(x+3)$

$$x^2+7x+11=A(x+2)(x+3)+B(x+3)+C(x+2)^2\rightarrow(\text{ii})$$

$$=A(x^2+5x+6)+B(x+3)+C(x^2+4x+4)$$

$$=Ax^2+5Ax+6A+Bx+3B+Cx^2+4Cx+4C$$

$$=Ax^2+Cx^2+5Ax+Bx+4Cx+6A+3B+4C$$

$$=(A+C)x^2+(5A+B+4C)x+(6A+3B+4C)$$

By comparing coefficients of alike powers of 'x'

$1=A+C$ __________ (iii)

$7=5A+B+4C$ __________ (iv)

$11=6A+3B+4C$__________ (v)

Put $x+2\Rightarrow x=-2$ in equation _______ (ii)

$$(-2)^2+7(-2)+11=A(-2+2)(-2+3)$$
$$+B(-2+3)+C(-2+2)^2$$

$$4-14+11=A(0)(1)+B(1)+C(0)^2-10+11=B$$

$$1=B$$

Or $B=1$

Put $x+3=0\Rightarrow x=-3$ in equation (ii)

$$(-3)^2+7(-3)+11=A(-3+2)(-3+3)$$
$$+B(-3+3)+C(-3+2)^2$$

$$9-21+11=A(-1)(0)+B(0)+C(-1)^2$$

$-1=0+0+C$

$C=-1$

Put C = -1 in equation (iii)

$1=A+(-1)$

$1+1=A$

Or A = 2

Now putting the values of A,B,C in (i)

$$\frac{x^2+7x+11}{(x+2)^2(x+3)}=\frac{2}{x+2}+\frac{1}{(x+2)^2}+\frac{-1}{x+3}$$

$$=\frac{2}{x+2}+\frac{1}{(x+2)^2}-\frac{1}{x+3}$$

Q.3 $\frac{9}{(x-1)(x+2)^2}$ **(A.B)**

(SWL 2014, RWP 2017, SGD 2015)

Solution:

Let $\frac{9}{(x-1)(x+2)^2}=\frac{A}{x-1}+\frac{B}{x+2}+\frac{C}{(x+2)^2}\rightarrow(\text{i})$

Multiplying by $(x-1)(x+2)^2$

$$9=A(x+2)^2+B(x-1)(x+2)+C(x-1)\rightarrow(\text{ii})$$

$$9=A(x^2+4x+4)+B(x^2+x-2)+C(x-1)$$

$$0x^2+0x+9=Ax^2+4Ax+4A+Bx^2+Bx-2B+Cx-C$$

$$0x^2+0x+9=Ax^2+Bx^2+4Ax+Bx$$
$$+Cx+4A-2B-C$$

$$0x^2+0x+9=(A+B)x^2+(4A+B+C)x$$
$$+(4A-2B-C)$$

By comparing coefficients of alike powers of 'x'

$0=A+B\rightarrow$ (iii)

$0=4A+B+C\rightarrow$ (iv)

$9=4A-2B-C\rightarrow$ (v)

Put $x-1=0\Rightarrow x=1$ in equation (ii)

$$9=A(1+2)^2+B(1-1)(1+2)+C(1-1)$$

$$9=A(3)^2+B(0)+C(0)$$

$9=9A+0+0$

$9=9A$

Or A = 1

Put $x+2=0\Rightarrow x=-2$ in equation (ii)

$$9=A(-2+2)+B(-2-1)(-2+2)+C(-2-1)$$

$$9=A(0)+B(-3)(0)+C(-3)$$

$$9=0+0-3C$$

$$C=-\frac{9}{3}$$

$$C=-3$$

Put A = 1 in equation (iii)

$0 = 1 + B$

Now putting the values in equation (i)

$$\Rightarrow \frac{9}{(x-1)(x+2)^2} = \frac{1}{x-1} + \frac{-1}{x+2} + \frac{-3}{(x+2)^2}$$

$$= \frac{1}{x-1} - \frac{1}{x+2} - \frac{3}{(x+2)^2}$$

Q.4 $\frac{x^4+1}{x^2(x-1)}$ **(K.B + A.B)**

Solution:

$$\frac{x^4+1}{x^2(x-1)} = \frac{x^4+1}{x^3-x^2} \quad \text{(improper fraction)}$$

$$\begin{array}{r|l} & x+1 \\ x^3-x^2 & x^4+1 \\ & \pm x^4 \mp x^3 \\ \hline & x^3+1 \\ & \pm x^3 \mp x^2 \\ \hline & x^2+1 \end{array}$$

$$\frac{x^4+1}{x^2(x-1)} = x+1+\frac{x^2+1}{x^3-x^2}$$

$$= x+1+\frac{x^2+1}{x^2(x-1)} \rightarrow \text{(i)}$$

Let $\frac{x^2+1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} \rightarrow \text{(ii)}$

$$x^2+1 = Ax(x-1) + B(x-1) + C(x^2)$$

$$x^2+1 = Ax^2 - Ax + Bx - B + Cx^2$$

$$x^2+0x+1 = Ax^2 + Cx^2 + Bx - Ax - B$$

$$= (A+C)x^2 + (B-A)x - B$$

By comparing coefficients of alike powers of 'x'

$1 = A + C \rightarrow$ (iii)

$0 = B - A \rightarrow$ (iv)

$1 = -B \rightarrow$ (v)

From equation (v)

$B = -1$

Put in equation (iv)

$0 = -1 - A$

A = -1

Put in equation (iii)

$1 = -1 + C$

$1 + 1 = C$

$C = 2$

Now putting the values in equation (ii)

$$\frac{x^2+1}{x^2(x-1)} = \frac{-1}{x} + \frac{-1}{x^2} + \frac{2}{x-1}$$

$$= -\frac{1}{x} - \frac{1}{x^2} + \frac{2}{x-1}$$

Now putting the values in equation (i)

$$\Rightarrow \frac{x^4+1}{x^2(x-1)} = x+1-\frac{1}{x}-\frac{1}{x^2}+\frac{2}{x-1}$$

Q.5 $\frac{7x+4}{(3x+2)(x+1)^2}$ **(A.B)**

Solution:

Let $\frac{7x+4}{(3x+2)(x+1)^2} = \frac{A}{3x+2} + \frac{B}{x+1} + \frac{C}{(x+1)^2} \rightarrow \text{(i)}$

Multiplying by $(3x+2)(x+1)^2$

$$7x+4 = A(x+1)^2 + B(x+1)(3x+2) + C(3x+2) \rightarrow \text{(ii)}$$

$$0x^2+7x+4 = A(x^2+2x+1) + B(3x^2+3x+2) + 3Cx + 2C$$

$$= Ax^2 + 2Ax + A + 3Bx^2 + 3Bx + 2B + 3Cx + 2C$$

$$= Ax^2 + 3Bx^2 + 2Ax + 3Bx + 3Cx + A + 2B + 2C$$

$$0x^2+7x+4 = (A+3B)x^2 + (2A+3B+3C)x + (A+2B+2C)$$

By Comparing coefficients of alike powers of x

$A = ?$

$0 = A + 3B \rightarrow$ (iii)

$7 = 2A + 3B + 3C \rightarrow$ (iv)

$4 = A + 2B + 2C \rightarrow$ (v)

Put $x+1 \Rightarrow 0$, $x = -1$ in equation (ii)

$$7(-1)+4 = A(-1+1)^2 + B(-1+1) + C[3(-1)+2]$$

$$-7+4 = 0+0+C(-3+2)$$

$$-3 = (-1)C$$

Or C = 3

Put A = –6 in equation (iii)

$0 = -6 + 3B$

$6 = 3B$

Or $B = 2$

Putting all the values in equation (i)

$$\frac{7x+4}{(3x+2)(x+1)^2}=\frac{-6}{3x+2}+\frac{2}{x+1}+\frac{3}{(x+1)^2}$$

Q.6 $\frac{1}{(x-1)^2(x+1)}$ **(A.B)**

Solution:

Let $\frac{1}{(x-1)^2(x+1)}=\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{x+1}\rightarrow(\text{i})$

Multiplication by $(x-1)^2(x+1)$

$1=A(x-1)(x+1)+B(x+1)+C(x-1)^2\rightarrow(\text{ii})$

$0x^2+0x+1=A(x-1)(x+1)+B(x+1)+C(x-1)^2$

$0x^2+0x+1=A(x^2-1)+B(x+1)+C(x^2-2x+1)$

$=Ax^2+Cx^2+Bx-2Cx-A+B+C$

$=(A+C)x^2+(B-2C)x+(-A+B+C)$

Comparing coefficients of powers $'x'$

$0=A+C$ __________ (iii)

$0=B-2C$ __________ (iv)

$1=-A+B+C$ __________ (v)

Put $x-1=0\Rightarrow x=1$ in equation (ii)

$1=A(0)(1+1)+B(1+1)+C(0)^2$

$1=0+B(2)+C(0)$

$1=2B$

$Or\quad B=\frac{1}{2}$

Put $B=\frac{1}{2}$ in equation (iv)

$0=\frac{1}{2}-2C$

$2C=\frac{1}{2}$

$C=\frac{1}{4}$

Put $C=\frac{1}{4}$ in equation (iii)

$0=A+\frac{1}{4}$

$A=-\frac{1}{4}$

Now putting the values in equation (i)

$$\frac{1}{(x-1)^2(x+1)}=\frac{-1/4}{x-1}+\frac{1/2}{(x-1)^2}+\frac{1/4}{x+1}$$

$$\frac{1}{(x-1)^2(x+1)}=\frac{-1}{4(x-1)}+\frac{1}{2(x-1)^2}+\frac{1}{4(x+1)}$$

Q.7 $\frac{3x^2+15x+16}{(x+2)^2}$ **(A.B)**

Solution:

$$\frac{3x^2+15x+16}{(x+2)^2}=\frac{3x^2+15x+16}{x^2+4x+4}$$

(improper fraction)

$$\begin{array}{r|l} & 3 \\ x^2+4x+4 & 3x^2+15x+16 \\ & \pm3x^2\pm12x\pm12 \\ \hline & 3x+4 \end{array}$$

$\frac{3x^2+15x+16}{(x+2)^2}=3+\frac{3x+4}{x^2+4x+4}$ ______ (i)

Let $\frac{3x+4}{(x+2)^2}=\frac{A}{x+2}+\frac{B}{(x+2)^2}$ _______ (ii)

Multiplying by $(x+2)^2$

$3x+4=A(x+2)+B$

$3x+4=Ax+2A+B$

By comparing coefficients of powers of $'x'$

3 = A __________ (iii)

$4=2A+B$ __________ (iv)

From equation (iii)

A = 3

Put in equation (iv)

$4=2(3)+B$

$4=6+B$

$4-6=B$

$B=-2$

Now putting values in equation (ii)

$$\frac{3x+4}{(x+2)^2}=\frac{3}{x+2}+\frac{-2}{(x+2)^2}$$

$$=\frac{3}{x+2}-\frac{2}{(x+2)^2}$$

Now putting the values in equation (i)

$$\Rightarrow \frac{3x^2+15x+16}{(x+2)^2}=3+\frac{3}{x+2}-\frac{2}{(x+2)^2}$$

Q.8 $\frac{1}{(x^2-1)(x+1)}$ **(K.B + A.B)**

Solution:

$$\frac{1}{(x^2-1)(x+1)}=\frac{1}{(x+1)(x-1)(x+1)}$$

$$=\frac{1}{(x+1)^2(x-1)}$$

Let $\frac{1}{(x-1)(x+1)^2}=\frac{A}{x-1}+\frac{B}{(x+1)}+\frac{C}{(x+1)^2}\to(\text{i})$

Multiplication by $(x-1)(x+1)^2$

$1=A(x+1)+B(x-1)(x+1)+C(x-1)\to(\text{ii})$

Put $x-1=0\Rightarrow x=1$ in equation (ii)

$1=A(1+1)^2+B(0)+C(0)$

$1=A(2)^2$

$1=4A$

$\Rightarrow A=\frac{1}{4}$

Put $x+1=0\Rightarrow x=-1$ in equation (ii)

$1=A(-1+1)+B(-1-1)(-1+1)+C(-1-1)$

$1=A(0)+B(0)+C(-2)$

$1=-2C$

$\Rightarrow C=-\frac{1}{2}$

Equation (ii)

$0x^2+0x+1=A(x+1)^2+B(x-1)(x+1)+C(x-1)$

$=A(x^2+2x+1)+B(x^2-1)+Cx-C$

$=Ax^2+2Ax+A+Bx^2-B+Cx-C$

$=Ax^2+Bx^2+2Ax+Cx+A-B-C$

$=(A+B)x^2+(2A+C)x+(A-B-C)$

By comparing coefficients of alike powers of x,

$0=A+B$ __________ (iii)

$0=2A+C$ __________ (iv)

$1=A-B-C$ __________ (v)

Put $A=\frac{1}{4}$ in equation (iii)

$0=\frac{1}{4}+B$

$B=-\frac{1}{4}$

Now putting values in equation (i)

$$\frac{1}{(x-1)(x+1)^2}=\frac{1/4}{x-1}+\frac{-1/4}{x+1}+\frac{-1/2}{(x+1)^2}$$

$$\Rightarrow\frac{1}{(x-1)(x+1)^2}=\frac{1}{4(x-1)}-\frac{1}{4(x+1)}-\frac{1}{2(x+1)^2}$$