## Factorization of the Expression of the Types

$a^4 + a^2b^2 + b^4$ or $a^4 + 4b^4$

(K.B)

**Example # 1** (A.B)

**Factorize**: $81x^4 + 36x^2y^2 + 16y^4$

**Solution:**

$81x^4 + 36x^2y^2 + 16y^4$

$= (9x^2)^2 + 72x^2y^2 + (4y^2)^2 - 36x^2y^2$

$\because (a+b)^2 = a^2 + 2ab + b^2$

$= (9x^2 + 4y^2)^2 - (6xy)^2$

$\because a^2 - b^2 = (a+b)(a-b)$

$= [9x^2 + 4y^2 + 6xy][9x^2 - 6xy + 4y^2]$

**Example # 2** (A.B)

**Factorize**: $9x^4 + 3y^4$

**Solution:**

$9x^4 + 36y^4$

$= 9x^4 + 36y^4 + 36x^2y^2 - 36x^2y^2$

$= (3x^2)^2 + 2(3x^2)(6y^2) + (6y^2)^2 - (6xy)^2$

$\because (a+b)^2 = a^2 + 2ab + b^2$

$= [3x^2 + 6y^2]^2 - (6xy)^2$

$\because a^2 - b^2 = (a+b)(a-b)$

$= [3x^2 + 6y^2 - 6xy][3x^2 + 6y^2 + 6xy]$

$= [3x^2 - 6xy + 6y^2][3x^2 + 6xy + 6y^2]$

## Factorization of the Expression of the Type $x^2 + px + q$ (K.B)

**Example**

**Factorize:** (A.B)

(i) $x^2 - 7x + 12$

(ii) $x^2 + 5x - 36$

**(i)** $x^2 - 7x + 12$

$\because (-3) + (-4) = -7$ and $(-3)(-4) = 12$

Hence

$x^2 - 7x + 12 = x^2 - 3x - 4x + 12$

$= x(x-3) - 4(x-3)$

$= (x-3)(x-4)$

**(ii)** $x^2 + 5x - 36$

$\because 9 + (-4) = 5$ and $9 \times (-4) = -36$

Hence

$x^2 + 5x - 36 = x^2 + 9x - 4x - 36$

$= x(x+9) - 4(x+9)$

$= (x+9)(x-4)$

## Factorization of the Expression of the Type $ax^2 + bx + c,\ a \neq 0$:

**Example**

**Factorize:** $9x^2 + 21x - 8$

**Solution:**

$\because ac = (9)(-8) = -72$ and

$24 + (-3) = 21$

Hence

$9x^2 + 21x - 8$

$= 9x^2 + 24x - 3x - 8$

$= 3x(3x+8) - (3x+8)$

$= (3x+8)(3x-1)$

## Factorization of the Expressions of the Types (K.B)

$(ax^2 + bx + c)(ax^2 + bx + d) + k$

$(x+a)(x+b)(x+c)(x+d) + k$

$(x+a)(x+b)(x+c)(x+d) + kx^2$

**Example # 1** **(A.B)**

**Factorize:**

$$\left(x^2-4x-5\right)\left(x^2-4x-12\right)-144$$

**Solution:**

$$\left(x^2-4x-5\right)\left(x^2-4x-12\right)-144$$

Let $y=x^2-4x$

Then

$$\left(x^2-4x-5\right)\left(x^2-4x-12\right)-144$$

$$=(y-5)(y-12)-144$$

$$=y^2-17y-84$$

$$=y^2-21y+4y-84$$

$$=y(y-21)+4(y-21)$$

$$=(y-21)(y+4)$$

Putting the value of $y$

$$=\left(x^2-4x-21\right)\left(x^2-4x+4\right)$$

$$=\left(x^2-7x+3x-21\right)\left(x^2-2(x)(2)+2^2\right)$$

$$\because a^2-2ab+b^2=(a-b)^2$$

$$=\left[x(x-7)+3(x-7)\right]\left[x-2\right]^2$$

$$=(x-7)(x+3)(x-2)(x-2)$$

**Example # 2** **(A.B)**

**Factorize:**

$$(x+1)(x+2)(x+3)(x+4)-120$$

**Solution:**

We observe that $1+4=2+3$.

$$\therefore (x+1)(x+2)(x+3)(x+4)-120$$

$$=\left[(x+1)(x+4)\right]\left[(x+2)(x+3)\right]-120$$

$$=\left(x^2+5x+4\right)\left(x^2+5x+6\right)-120$$

Put $x^2+5x=y$

$$=(y+4)(y+6)-120$$

$$=y^2+10y+24-120$$

$$=y^2+10y-96$$

$$=y^2+16y-6y-96$$

$$=y(y+16)-6(y+16)$$

$$=(y+16)(y-6)$$

Putting the value of $y$

$$=\left(x^2+5x+16\right)\left(x^2+5x-6\right)$$

$$=\left(x^2+5x+16\right)\left(x^2+6x-x-6\right)$$

$$=\left(x^2+5x+16\right)\left(x(x+6)-1(x-6)\right)$$

$$=\left(x^2+5x+16\right)(x+6)(x-1)$$

**Example # 3** **(A.B)**

**Factorize:** $\left(x^2-5x+6\right)\left(x^2+5x+6\right)-2x^2$

**Solution:**

$$\left(x^2-5x+6\right)\left(x^2+5x+6\right)-2x^2$$

$$=\left[x^2-3x-2x+6\right]\left[x^2+3x+2x+6\right]-2x^2$$

$$=\left[x(x-3)-2(x-3)\right]\left[x(x+3)+2(x+3)\right]-2x^2$$

$$=(x-3)(x-2)(x+3)(x+2)-2x^2$$

$$=(x-3)(x+3)(x-2)(x+2)-2x^2$$

$$=\left(x^2-9\right)\left(x^2-4\right)-2x^2$$

$$=x^4-13x^2+36-2x^2$$

$$=x^4-15x^2+36$$

$$=x^4-12x^2-3x^2+36$$

$$=x^2\left(x^2-12\right)-3\left(x^2-12\right)$$

$$=\left(x^2-12\right)\left(x^2-3\right)$$

$$=\left[(x)^2-2\left(\sqrt{3}\right)^2\right]\left[(x)^2-\left(\sqrt{3}\right)^2\right]$$

$$\because a^2-b^2=(a+b)(a-b)$$

$$=\left[x-2\sqrt{3}\right]\left[x+2\sqrt{3}\right]\left[x-\sqrt{3}\right]\left[x+\sqrt{3}\right]$$

## Factorization of Expressions of the Types $a^3+3a^2b+3ab^2+b^3$, $a^3-3a^2b+3ab^2-b^3$ **(K.B)**

**Example**

**Factorize:** $x^3-8y^3-6x^2y+12xy^2$

***Solution:*** **(A.B)**

$$x^3-8y^3-6x^2y+12xy^2$$

$$=x^3-(2y)^3-3(x)^2(2y)+3(x)\left(2y^2\right)$$

$$=x^3-3(x)^2(2y)+3(x)(2y)^2-(2y)^3$$

$$\because a^3-3a^2b+3ab^2-b^3=(a-b)^3$$

$$=(x-2y)^3$$

$$=(x-2y)(x-2y)(x-2y)$$

## Factorization of Expressions of the Types $a^3\pm b^3$

We recall the formulas.

$$a^3+b^3=(a+b)(a^2-ab+b^2)$$

$$a^3-b^3=(a-b)(a^2+ab+b^2)$$

**Example # 1**

**Factorize**: $27x^3+64y^3$

**Solution**:

$$27x^3+64y^3=(3x)^3+(14y)^3$$

$$\because a^3+b^3=(a+b)(a^2-ab+b^2)$$

$$=[3x+4y]\left[(3x)^2-(3x)(4y)+(4y)^2\right]$$

$$=[3x+4y]\left[9x^2-12xy+16y^2\right]$$

## Exercise 5.2

**Q.1 Factorize**

**(i)** $x^4+\dfrac{1}{x^4}-3$

**Solution:** **(A.B)**

$$x^4+\frac{1}{x^4}-3$$

$$=(x^2)^2+\left(\frac{1}{x^2}\right)^2-3$$

$$=\left[(x^2)^2+\left(\frac{1}{x^2}\right)^2-2\right]-1$$

$$\because a^2-2ab+b^2=(a-b)^2$$

$$=\left(x^2-\frac{1}{x^2}\right)^2-(1)^2$$

$$\because a^2-b^2=(a+b)(a-b)$$

$$=\left(x^2-\frac{1}{x^2}+1\right)\left(x^2-\frac{1}{x^2}-1\right)$$

**(ii)** $3x^4+12y^4$

**Solution:**

$$3x^4+12y^4$$

$$=3(x^4+4y^4)$$

By adding and subtracting by $2(x^2)(2y^2)$

$$=3\left[(x^2)^2+(2y^2)^2+2(x^2)(2y^2)-2(x^2)(2y^2)\right]$$

$$\because a^2+2ab+b^2=(a+b)^2$$

$$=3\left[(x^2+2y^2)^2-4x^2y^2\right]$$

$$=3\left[(x^2+2y^2)^2-(2xy)^2\right]$$

$$\because a^2-b^2=(a+b)(a-b)$$

$$=3\left[(x^2+2y^2+2xy)(x^2+2y^2-2xy)\right]$$

$$=3\left[(x^2+2xy+2y^2)(x^2-2xy+2y^2)\right]$$

**(iii)** $a^4+3a^2b^2+4b^4$

**Solution:**

$$a^4+3a^2b^2+4b^4$$

$$=(a^4+4b^4)+3a^2b^2$$

$$=(a^2)^2+(2b^2)^2+3a^2b^2$$

By adding and subtracting by $2(a^2)(2b^2)$

$$=(a^2)^2+(2b^2)^2+2(a^2)(2b^2)-2(a^2)(2b^2)+3a^2b^2$$

$$=\left[(a^2)^2+(2b^2)^2+2(a^2)(2b^2)\right]-2(a^2)(2b^2)+3a^2b^2$$

$$\because a^2+2ab+b^2=(a+b)^2$$

$$=(a^2+2b^2)^2-a^2b^2$$

$$=(a^2+2b^2)^2-(ab)^2$$

$$\because a^2-b^2=(a+b)(a-b)$$

$$=(a^2+2b^2+ab)(a^2+2b^2-ab)$$

**(iv)** $4x^4+81$

**Solution:** **(A.B)**

$$4x^4+81$$

$$=(2x^2)^2+(9)^2$$

By adding and subtracting by $2(2x^2)(9)$

$$=\left[(2x^2)^2+(9)^2+2(2x^2)(9)-2(2x^2)(9)\right]$$

$$=\left[(2x^2)^2+(9)^2+2(2x^2)(9)\right]-2(2x^2)(9)$$

$$\because a^2+2ab+b^2=(a+b)^2$$

$$=(2x^2+9)^2-36x^2$$

$$=(2x^2+9)^2-(6x)^2$$

$$\because a^2-b^2=(a+b)(a-b)$$

$$=(2x^2+9+6x)(2x^2+9-6x)$$

$=\left(2x^2+6x+9\right)\left(2x^2-6x+9\right)$

**(v)** $x^4+x^2+25$ **(MTN 2016)**

**Solution:**

$x^4+x^2+25$

$=\left(x^4+25\right)+x^2$

$=\left[\left(x^2\right)^2+(5)^2\right]+x^2$

By adding and subtracting by $2\left(x^2\right)(5)$

$=\left[\left(x^2\right)^2+(5)^2+2\left(x^2\right)(5)-2\left(x^2\right)(5)\right]+x^2$

$=\left[\left(x^2\right)^2+(5)^2+2\left(x^2\right)(5)\right]-2\left(x^2\right)(5)+x^2$

$\because a^2+2ab+b^2=\left(a+b\right)^2$

$=\left(x^2+5\right)^2-10x^2+x^2$

$=(x^2+5)^2-9x^2$

$=(x^2+5)^2-(3x)^2$

$\because a^2-b^2=(a+b)(a-b)$

$=\left(x^2+5+3x\right)\left(x^2+5-3x\right)$

$=(x^2+3x+5)(x^2-3x+5)$

**(vi)** $x^4+4x^2+16$

**Solution:**

$x^4+4x^2+16$

$=(x^2)^2+16+4x^2$

$=(x^2)^2+(4)^2+4x^2$

By adding and subtracting by $2\left(x^2\right)(4)$

$=\left(x^2\right)^2+\left(4\right)^2+2\left(x^2\right)(4)-2\left(x^2\right)(4)+4x^2$

$=(x^2)^2+(4)^2+2(x^2)(4)-2(x^2)(4)+4x^2$

$\because a^2+2ab+b^2=\left(a+b\right)^2$

$=\left(x^2+4\right)^2-8x^2+4x^2$

$=(x^2+4)^2-4x^2$

$=(x^2+4)^2-(2x)^2$

$\because a^2-b^2=(a+b)(a-b)$

$=\left(x^2+4+2x\right)\left(x^2+4-2x\right)$

$=\left(x^2+2x+4\right)\left(x^2-2x+4\right)$

**Q.2**

**(i)** $x^2+14x+48$

**Solution:**

$x^2+14x+48$

$=x^2+8x+6x+48$

$=x(x+8)+6(x+8)$

$=(x+8)(x+6)$

**(ii)** $x^2-21x+108$

**Solution:**

$x^2-21x+108$

$=x^2-12x-9x+108$

$=x(x-12)-9(x-12)$

$=(x-9)(x-12)$

**(iii)** $x^2-11x-42$

**Solution:**

$x^2-11x-42$

$=x^2-14x+3x-42$

$=x(x-14)+3(x-14)$

$=(x+3)(x-14)$

**(iv)** $x^2+x-132$

**Solution:**

$x^2+x-132$

$=x^2+12x-11x-132$

$=x(x+12)-11(x+12)$

$=(x-11)(x+12)$

**Q.3**

**(i)** $4x^2+12x+5$

**Solution:** **(A.B)**

$4x^2+12x+5$

$=4x^2+2x+10x+5$

$=2x(2x+1)+5(2x+1)$

$=(2x+5)(2x+1)$

**(ii)** $30x^2+7x-15$ **(LHR 2014)**

**Solution:**

$30x^2+7x-15$

$=30x^2+25x-18x-15$

$= 5x(6x+5) - 3(6x+5)$
$= (5x-3)(6x+5)$

**(iii)** $24x^2 - 65x + 21$

**Solution:**

$24x^2 - 65x + 21$
$= 24x^2 - 56x - 9x + 21$
$= 8x(3x-7) - 3(3x-7)$
$= (8x-3)(3x-7)$

**(iv)** $5x^2 - 16x - 21$

**Solution:**

$5x^2 - 16x - 21$
$= 5x^2 + 5x - 21x - 21$
$= 5x(x+1) - 21(x+1)$
$= (5x-21)(x+1)$

**(v)** $4x^2 - 17xy + 4y^2$

**Solution:**

$4x^2 - 17xy + 4y^2$
$= 4x^2 - 16xy - xy + 4y^2$
$= 4x(x-4y) - y(x-4y)$
$= (4x-y)(x-4y)$

**(vi)** $3x^2 - 38xy - 13y^2$

**Solution:**

$3x^2 - 38xy - 13y^2$
$= 3x^2 - 39xy + xy - 13y^2$
$= 3x(x-13y) + y(x-13y)$
$= (3x+y)(x-13y)$

**(vii)** $5x^2 + 33xy - 14y^2$

**Solution:**

$5x^2 + 33xy - 14y^2$
$= 5x^2 + 35xy - 2xy - 14y^2$
$= 5x(x+7y) - 2y(x+7y)$
$= (5x-2y)(x+7y)$

**(viii)** $\left(5x-\frac{1}{x}\right)^2 + 4\left(5x-\frac{1}{x}\right) + 4, x \neq 0$

**Solution:**

$\left(5x-\frac{1}{x}\right)^2 + 4\left(5x-\frac{1}{x}\right) + 4$

$= \left(5x-\frac{1}{x}\right)^2 + 2\left(5x-\frac{1}{x}\right)(2) + (2)^2$

$\because a^2 + 2ab + b^2 = (a+b)^2$

$= \left(5x-\frac{1}{x}+2\right)^2$

$= \left(5x-\frac{1}{x}+2\right)\left(5x-\frac{1}{x}+2\right)$

**Q.4**

**(i)** $(x^2+5x+4)(x^2+5x+6)-3$

**Solution:** $(x^2+5x+4)(x^2+5x+6)-3$

Suppose that
$x^2 + 5x = y$
**So,**
$(x^2+5x+4)(x^2+5x+6)-3$
$= (y+4)(y+6) - 3$
$= [y(y+6) + 4(y+6) - 3$
$= (y^2 + 6y + 4y + 24) - 3$
$= (y^2 + 10y + 24) - 3$
$= y^2 + 10y + 24 - 3$
$= y^2 + 10y + 21$
$= y^2 + 7y + 3y + 21$
$= y(y+7) + 3(y+7)$
$= (y+3)(y+7)$
We know that $y = x^2 + 5x$
$= (x^2+5x+3)(x^2+5x+7)$

**(ii)** $(x^2-4x)(x^2-4x-1)-20$

**Solution:**

$(x^2-4x)(x^2-4x-1)-20$
Suppose that
$x^2 - 4x = y$
So, $(x^2-4x)(x^2-4x-1)-20$
$= (y)(y-1) - 20$
$= (y^2 - y) - 20$
$= y^2 - y - 20$
$= y^2 - 5y + 4y - 20$
$= y(y-5) + 4(y-5)$

$= (y+4)(y-5)$

Putting the value of $y$

$= (x^2-4x+4)(x^2-4x-5)$

$= \left[(x)^2-2(x)(2)+(2)^2\right]\left[x^2-5x+x-5\right]$

$= (x-2)^2\left[x(x-5)+1(x-5)\right]$

$= (x-2)^2(x-5)(x+1)$

$= (x-5)(x+1)(x-2)^2$

**(iii)** $(x+2)(x+3)(x+4)(x+5)-15$

**Solution:**

$(x+2)(x+3)(x+4)(x+5)-15$

$= [(x+2)(x+5)][(x+3)(x+4)]-15$

$= [x(x+5)+2(x+5)][x(x+4)+3(x+4)]-15$

$= [x^2+5x+2x+10][x^2+4x+3x+12]-15$

$= (x^2+7x+10)(x^2+7x+12)-15$

Put $x^2+7x=y$

So,

$(x^2+7x+10)(x^2+7x+12)-15$

$= (y+10)(y+12)-15$

$= [y(y+12)+10(y+12)]-15$

$= (y^2+12y+10y+120)-15$

$= (y^2+22y+120)-15$

$= y^2+22y+120-15$

$= y^2+22y+105$

$= y^2+15y+7y+105$

$= y(y+15)+7(y+15)$

$= y(y+15)+7(y+15)$

$= (y+7)(y+15)$

Putting the value of $y$

$= (x^2+7x+7)(x^2+7x+15)$

**(iv)** $(x+4)(x-5)(x+6)(x-7)-504$

**Solution:**

$(x+4)(x-5)(x+6)(x-7)-504$

$= [(x+4)(x-5)][(x+6)(x-7)]-504$

$= [x(x-5)+4(x-5)][x(x-7)+6(x-7)]-504$

$= (x^2-5x+4x-20)(x^2-7x+6x-42)-504$

$= (x^2-x-20)(x^2-x-42)-504$

Suppose that

$x^2-x=y$

So,

$= (y-20)(y-42)-504$

$= [y(y-42)-20(y-42)]-504$

$= (y^2-42y-20y+840)-504$

$= y^2-62y+840-504$

$= y^2-62y+336$

$= y^2-56y-6y+336$

$= y(y-56)-6(y-56)$

$= (y-6)(y-56)$

We know that $a=x^2-x$

$= (x^2-x-6)(x^2-x-56)$

$= (x^2-3x+2x-6)(x^2-8x+7x-56)$

$= [x(x-3)+2(x-3)][x(x-8)+7(x-8)]$

$= (x+2)(x-3)(x+7)(x-8)$

**(v)** $(x+1)(x+2)(x+3)(x+6)-3x^2$

**Solution:**

$(x+1)(x+2)(x+3)(x+6)-3x^2$

$= [(x+1)(x+6)][(x+2)(x+3)]-3x^2$

$= [x(x+6)+1(x+6)][x(x+3)+2(x+3)]-3x^2$

$= (x^2+6x+x+6)(x^2+3x+2x+6)-3x^2$

$= (x^2+6+7x)(x^2+6+5x)-3x^2$

Suppose that

$x^2+6=y$

So,

$= (y+7x)(y+5x)-3x^2$

$= [y(y+5x)+7x(y+5x)]-3x^2$

$= (y^2+5xy+7xy+35x^2-3x^2)$

$= y^2+12xy+32x^2$

$= y^2+8xy+4xy+32x^2$

$= y(y+8x)+4x(y+8x)$

$= (y+4x)(y+8)$

We know that $y=x^2+6$

$= (x^2+6+4x)(x^2+6+8x)$

$= (x^2+4x+6)(x^2+8x+6)$

**Q.5**

**(i)** $x^3+48x-12x^2-64$

**Solution:**

$$x^3+48x-12x^2-64$$
$$=x^3-12x^2+48x-64$$
$$\because a^3-3a^2b+3ab^2-b^3=(a-b)^3$$
$$=(x)^3-3(x)^2(4)+3(x)(4)^2-(4)^3$$
$$=(x-4)^3$$

**(ii)** $8x^3+60x^2+150x+125$

**Solution:**

$$8x^3+60x^2+150x+125$$
$$=(2x)^3+3(2x)^2(5)+3(2x)(5)^2+(5)^3$$
$$\because a^3+3a^2b+3ab^2+b^3=(a+b)^3$$
$$=(2x+5)^3$$

**(iii)** $x^3-18x^2+108x-216$

**Solution:**

$$x^3-18x^2+108x-216$$
$$=(x)^3-3(x)^2(6)+3(x)(6)^2-(6)^3$$
$$\because a^3-3a^2b+3ab^2-b^3=(a-b)^3$$
$$=(x-6)^3$$

**(iv)** $8x^3-125y^3-60x^2y+150xy^2$

**Solution:**

$$8x^3-125y^3-60x^2y+150xy^2$$
$$=8x^3-60x^2y+150xy^2-125y^3$$
$$=(2x)^3-3(2x)^2(5y)+3(2x)(5y)^2-(5y)^3$$
$$\because a^3-3a^2b+3ab^2-b^3=(a-b)^3$$
$$=(2x-5y)^3$$

**Q.6**

**(i)** $27+8x^3$

**(GRW 2017, SWL 2014, 15, MTN 2015, SGD 2013)**

**Solution:**

$$27+8x^3$$
$$=(3)^3+(2x)^3$$
$$\because a^3+b^3=(a+b)(a^2-ab+b^2)$$
$$=(3+2x)[(3)^2-(3)(2x)+(2x)^2]$$
$$=(3+2x)(9-6x+4x^2)$$

**(ii)** $125x^3-216y^3$ **(SWL2013, D.GK 2017)**

**Solution:**

$$125x^3-216y^3$$
$$=(5x)^3-(6y)^3$$
$$\because a^3-b^3=(a-b)(a^2+ab+b^2)$$
$$=(5x-6y)[(5x)^2+(5x)(6y)+(6y)^2]$$
$$=(5x-6y)(25x^2+30xy+36y^2)$$

**(iii)** $64x^3+27y^3$

**Solution:**

$$64x^3+27y^3$$
$$=(4x)^3+(3y)^3$$
$$\because a^3+b^3=(a+b)(a^2-ab+b^2)$$
$$=(4x+3y)\left[(4x)^2-(4x)(3y)+(3y)^2\right]$$
$$=(4x+3y)(16x^2-12xy+9y^2)$$

**(iv)** $8x^3+125y^3$

**Solution:**

$$8x^3+125y^3$$
$$=(2x)^3+(5y)^3$$
$$\because a^3+b^3=(a+b)(a^2-ab+b^2)$$
$$=(2x+5y)\left[(2x)^2-(2x)(5y)+(5y)^2\right]$$
$$=(2x+5y)(4x^2-10xy+25y^2)$$