Mathematics-9

Unit 5 – Exercise 5.3

REMAINDER THEORME AND FACTOR THEOREM

Remainder Theorem (K.B+U.B)

(LHR 2015, BWP 2017)

If a polynomial $p(x)$ is divided by a linear divisor $(x-a)$ until a constant remainder is obtained, then this remainder is equal to $p(a)$.

i.e. $R = P(a)$

Proof:

Let $q(x)$ be the quotient obtained after dividing $p(x)$ by $(x-a)$. As the divisor $(x-a)$ is linear, so the remainder must be a constant say R.

By division Algorithm we may write

$p(x)=(x-a)q(x)+R$

This is an identity in x and so is true for all real numbers x. In particular it is true for $x=a$. Therefore,

$p(a)=(a-a)q(a)+R=0+R$

i.e., $p(a)=R$.

Hence Proved

Note (K.B)

If the divisor is $(ax-b)$, we have

$p(x)=(ax-b)q(x)+R$

Substituting $ax-b=0$ or $x=\frac{b}{a}$, we obtain

$p\left(\frac{b}{a}\right)=0.\ q\left(\frac{b}{a}\right)+R=0+R=R$

Thus if the divisor is linear, the above theorem provides an efficient way of finding the remainder without being involved in the process of long division.

Example # 1 (A.B)

Find the remainder when $9x^2-6x+2$ is divided by

(i) $(x-3)$ **(ii)** $x+3$ **(iii)** $3x+1$

(iv) x

Solution:

Let $p(x)=9x^2-6x+2$

(i) Put $x-3=0$ or $x=3$ in $p(x)$

$p(3)=9(3)^2-6(3)+2=65$

$\therefore R=65$

(ii) Put $x+3=0$ or $x=-3$ in $p(x)$

$p(-3)=9(-3)^2-6(-3)+2=101$

$\therefore R=101$

(iii) Put $3x+1=0$ or $x=-\frac{1}{3}$ in $p(x)$

$P\left(-\frac{1}{3}\right)=9\left(-\frac{1}{3}\right)^2-6\left(-\frac{1}{3}\right)+2=5$

$\therefore R=5$

(iv) Put $x=0$ in $p(x)$

$P(0)=9(0)^2-6(0)+2=2$

$\therefore R=2$

Example # 2 (A.B)

Find the value of k if the expression x^3+kx^2+3x-4 leaves a remainder of -2 when divided by $x+2$.

Solution:

$p(x)=x^3+kx^2+3x-4$

Put $x+2=0$ or $x=-2$ in $p(x)$

$p(-2)=(-2)^3+k(-2)^2+3(-2)-4.$

$=-8+4k-6-4$

$= -18 + 4k$

By the given condition, we have

$P(-2) = -2$

$\Rightarrow 4k - 18 = -2$

$4k = -2 + 18$

$4k = 16$

$\Rightarrow k = 4$

Zero of the Polynomial **(K.B)**

If a specific number $x = a$ is substituted for the variable x in a polynomial $p(x)$ so that value $P(a)$ is a zero then $x = a$ is called a zero of the polynomial $P(x)$.

Factor Theorem **(U.B)**

If a polynomial $P(x)$ is divided by a binomial $(x-a)$ such that remainder is zero, then $(x-a)$ is called factor of $P(x)$.

Or

The polynomial $(x-a)$ is a factor of the polynomial $P(x)$ if and only if $P(a) = 0$.

Proof:

Let $q(x)$ be the quotient and R the remainder when a polynomial $P(x)$ is divided by $(x-a)$ then by division Algorithm,

$P(x) = (x-a)q(x) + R$

By the Remainder Theorem, $R = P(\text{a})$.

Hence $P(x) = (x-a)q(x) + P(a)$

(i) Now if $P(a) = 0$ then $P(x)$ then $P(x) = (x-a)q(x)$ i.e., $(x-a)$ is a factor of $P(x)$

(ii) Conversely, if $(x-a)$ is a factor of $P(x)$, then the remainder upon dividing $P(x)$ by $(x-a)$ must be zero i.e. $P(a) = 0$

This complete the proof.

Example # 1 **(A.B)**

Determine if $(x-2)$ is a factor of $x^3 - 4x^2 + 3x + 2$.

Solution:

$P(x) = x^3 - 4x^2 + 3x + 2$

Then the remainder for $(x-2)$ is

$P(2) = (2)^3 - 4(2)^2 + 3(2) + 2$

$= 8 - 16 + 6 + 2 = 0$

Since remainder is 0, $(x-2)$ is a factor of the polynomial $P(x)$.

Example # 2

Find a polynomial $p(x)$ of degree 3 that has 2, - 1 and 3 as zeros (i.e. roots).

Solution:

Since $x = 2, -1, 3$ are roots of $p(x)$, so by factor theorem $(x-2), (x+1)$ and $(x-3)$ are the factors of $p(x)$.

Thus,

$p(x) = a(x-2)(x+1)(x-3)$ where $a \in R$

$p(x) = a(x-2)(x^2 - 3x + x - 3)$

$p(x) = a(x-2)(x^2 - 2x - 3)$

$p(x) = a(x^3 - 2x^2 - 3x - 2x^2 + 4x + 6)$

$p(x) = a(x^3 - 4x^2 + x + 6)$

Is the required cubic polynomial.

Exercise 5.3

Q.1 Use the remainder theorem to find the remainder when

(i) $3x^3 - 10x^2 + 13x - 6$ **is divided by** $(x-2)$.

Solution:

$P(x) = 3x^3 - 10x^2 + 13x - 6$

Since $P(x)$ is divided by $(x-2)$.

$\therefore P(2) = R$

$P(2) = 3(2)^3 - 10(2)^2 + 13(2) - 6$

$= 3(2)^3 - 10(2)^2 + 13(2) - 6$

$= 24 - 40 + 26 - 6$

$R = 4$

Hence 4 is the remainder

(ii) $4x^3 - 4x + 3$ **is divided by** $(2x-1)$

Solution: **(A.B)**

$P(x) = 4x^3 - 4x + 3$

Since $P(x)$ is divided by $(2x-1)$

$\therefore R = P\left(\frac{1}{2}\right) \quad \because 2x - 1 = 0 \Rightarrow x = \frac{1}{2}$

$P\left(\frac{1}{2}\right) = 4\left[\frac{1}{2}\right]^3 - 4^2 \times \frac{1}{2} + 3$

$= 4 \times \frac{1}{8^2} - 2 + 3$

$= \frac{1}{2} - 2 + 3$

$= \frac{1-4+6}{2} = \frac{3}{2}$

$R = \frac{3}{2}$

Hence $\frac{3}{2}$ is the remainder

(iii) $6x^4 + 2x^3 - x + 2$ **is divided by** $(x+2)$ **from** $x + 2 = 0$

Solution: Given that

$P(x) = 6x^4 + 2x^3 - x + 2$

Since $P(x)$ is divided by $(x+2)$

$\therefore R = P(-2)$

$P(-2) = 6(-2)^4 + 2(-2)^3 - (-2) + 2$

$= 96 - 16 + 2 + 2$

$R = 84$

Hence 84 is the remainder.

(iv) $(2x-1)^3 + 6(3+4x)^2 - 10$ **is divided by** $2x + 1$.

Solution: Given that

$P(x) = (2x-1)^3 + 6(3+4x)^2 - 10$

Since $P(x)$ is divided by $2x+1$

$\therefore R = P\left(-\frac{1}{2}\right)$

$P\left(-\frac{1}{2}\right) = \left[2\left(-\frac{1}{2}\right) - 1\right]^3 + 6\left[3 + 4^2\left(\frac{-1}{2}\right)\right]^2 - 10$

$= [-1-1]^3 + 6[3-2]^2 - 10$

$= [-2]^3 + 6 - 10 = -8 + 6 - 10$

$R = -12$

Hence -12 is the remainder.

(v) $x^3 - 3x^2 + 4x - 14$ **is divided by** $(x+2)$ **from** $x + 2 = 0, x = -2$

Solution: Given that

$P(x) = x^3 - 3x^2 + 4x - 14$

Since $P(x)$ is divided by $(x+2)$

$\therefore R = P(-2)$

$P(-2) = (-2)^3 - 3(-2)^2 + 4(-2) - 14$

$= -8 - 12 - 8 - 14$

$R = -42$

Hence -42 is the remainder

Q.2

(i) **If $(x+2)$ is a factor of** $3x^2 - 4kx - 4k^2$ **then find the values of k** $x + 2 = 0 \; x = -2$

Solution: Given that **(A.B)**

$P(x) = 3x^2 - 4kx - 4k^2$

$P(-2) = 3(-2)^2 - 4k(-2) - 4k^2$

$P(-2) = 12 + 8k - 4k^2$

If $(x+2)$ is the factor then remainder is equal to zero

$P(-2) = 0$

$12 + 8k - 4k^2 = 0$

$-4(-3 - 2k + k^2) = 0$

$k^2 - 2k - 3 = 0 \quad \because -4 \neq 0$

$k^2 - 3k + k - 3 = 0$

$k(k-3) + 1(k-3) = 0$

$(k-3)(k+1) = 0$

Either

$k - 3 = 0$ or $k + 1 = 0$

$k = 3 \qquad k = -1$

(ii) **If $(x-1)$ is a factor of** $x^3 - kx^2 + 11x - 6$ **the find the value of k from** $x - 1 = 0 \; x = 1$

Solution: Given that

$P(x) = x^3 - kx^2 + 11x - 6$

$P(1) = (1)^3 - k(1)^2 + 11(1) - 6$

$P(1) = 1 - k + 11 - 6$

$P(1) = 6 - k$

If (x-1) is the factor then remainder is equal to zero

$P(1)=0$

$6-k=0$

$k=6$

Result:

$k=6$

Q.3 Without long division determine whether

(i) $(x-2)$ **and** $(x-3)$ **are factor of** $P(x)=x^3-12x^2+44x-48$

Solution:

Given that

$P(x)=x^3-12x^2+44x-48$

If (x-2) is the factor then remainder is equal to zero

$P(2)=(2)^3-12(2)^2+44(2)-48=8-48+88-48=0$

Hence $x-2$ is a factor of $P(x)$

For $x-3$

$R=P(3)$

$=(3)^3-12(3)^2+44(3)-48$

$=(3)^3-12(3)^2+44(3)-48$

$=27-108+132-48$

$=159-156$

$R=3\neq 0$

P(3) is not equal to zero then $x-3$ is not factor of $P(x)=x3-12x^2+44x-48$

(ii) $(x-2),(x+3)$ **and** $(x-4)$ **are factor of** $q(x)=x^3+2x^2-5x-6$

from $x-2=0, x=2$

Solution:

Given that

$q(x)=x^3+2x^2-5x-6$

Put $x-2=0 \Rightarrow x=2$

$R=q(2)$

$=(2)^3+2(2)^2-5(2)-6$

$R=8+8-10-6$

$R=16-16$

$R=0$

Hence $x-2$ is factor o f $q(x)=x^3+2x^2-5x-6$

Put $x+3=0 \Rightarrow x=-3$

$R=q(-3)$

$=(-3)^3+2(-3)^2-5(-3)-6$

$=-27+18+15-6$

$R=0$

Hence $x-2$ is factor of $q(x)=x^3+2x^2-5x-6$

Put $x-4=0 \Rightarrow x=4$

$R=q(4)$

$=(4)^3+2(4)^2-5(4)-6$

$=64+32-20-6$

$R=70\neq 0$

Hence $x-4$ is not a factor of $q(x)=x^3+2x^2-5x-6$

Q.4 For what value of 'm' is the polynomial P(x) $=4x^3-7x^2+6x-3m$ exactly divisible by $x+2$?

Solution:

$P(x)=4x^3-7x^2+6x-3m$

From $x+2=0$, $x=-2$

$P(-2)=4(-2)^3-7(-2)^2+6(-2)-3m$

$P(-2)=-32-28-12-3m=-72-3m$

If (x+2) is the factor then remainder is equal to zero

$P(-2)=0$

$-72-3m=0$

$-72=3m$

$m=-\dfrac{72}{3}$

$m=-24$

Result:

$m=-24$

Q.5 Determine the value of k **if** $P(x)=kx^3+4x^2+3x-4$ **and** $q(x)=x^3-4x+k$ **leaves the same remainder when divided by** $(x-3)$.

Solution:

$q(x)=x^3-4x+k$

Put x-3=0 or x=3

$R_1 = q(3)$

$= (3)^3 - 4(3) + k$

$= 27 - 12 + k$

$= 15 + k$

$R_1 = 15 + k$ ....(i)

$R_2 = P(3)$

$= k(3)^3 + 4(3)^2 + 3(3) - 4$

$= 27k + 36 + 9 - 4$

$R_2 = 27k + 41$ ....(ii)

Since it leaves the same remainder.

Hence $R_1 = R_2$

$15 + k = 27k + 41$

$15 - 41 = 27k - k$

$-26 = 26k$

$k = \frac{-26}{26}$

$k = -1$

Q.6 The remainder after dividing the polynomial $P(x) = x^3 + ax^2 + 7$ by $(x+1)$ is $2b$ calculate the value of a and b if this expression leaves a remainder of $(b+5)$ on being dividing by $(x-2)$

Solution:

Let

$P(x) = x^3 + ax^2 + 7$

Since $P(x)$ is divided by $(x+1)$

Put $x + 1 = 0$ $x = -1$

$R = P(-1)$

$= (-1)^3 + a(-1)^2 + 7$

$= -1 + a + 7$

$R = a + 6$

According to first condition remainder is 2b

$2b = a + 6$ …(i)

Since $P(x)$ is divided by $(x$-2)

Put $x - 2 = 0$ or $x = 2$

$P(2) = (2)^3 + a(2)^2 + 7$

$= 8 + 4a + 7$

$R = 15 + 4a$

According to second condition remainder is (b+5)

$15+4a = b+5$

$4a - b = 5 - 15$

$4a - b = -10$ ……. (ii)

Solving equations (i) and (ii)

From equation (ii) $b = 10 + 4a$

Putting the value of b in equation (i)

$a + 6 = 2(10 + 4a)$

$a = 20 + 8a - 6$

$-8a + a = 14$

$-7a = 14$

$a = \frac{14}{-7}$

$a = -2$

Putting the value of a in equation (ii)

$4a - b = -10$

$4(-2) - b = -10$

$-8 - b = -10$

$-8 + 10 = b$

$2 = b$

$b = 2$

Result:

$a = -2,\ b = 2$

Q.7 The polynomial $x^3 + lx^2 + mx + 24$ has a factor $(x+4)$ and it leaves a remainder of 36 when divided by $(x-2)$

Find the values of *l* and *m*.

Solution: **(K.B+U.B)**

Let

$P(x) = x^3 + lx^2 + mx + 24$

From $x + 4 = 0$ or $x = -4$

$P(-4) = (-4)^3 + l(-4)^2 + m(-4) + 24$

$P(-4) = -64 + 16l - 4m + 24$

$P(-4) = 16l - 4m - 40$

According to condition (x+4) is the factor then

$16l - 4m - 40 = 0$

$4[4l - m - 10] = 0$

$4l - m - 10 = 0$ (i)

from $x - 2 = 0$ or $x = 2$

Now $P(2) = (2)^3 + l(2)^2 + m(2) + 24$

$P(2) = 8 + 4l + 2m + 24$

$P(2) = 4l + 2m + 32$

According the condition

$4l + 2m + 32 = 36$

$4l + 2m = 36 - 32$

$4l + 2m = 4$

$4l + 2m - 4 = 0$ (ii)

Subtracting (i) from (ii)

$\cancel{4l} + 2m - 4 = 0$

$\pm\cancel{4l} \mp m \mp 10 = 0$

$3m + 6 = 0$

$3m = -6$

$m = \frac{-\cancel{6}2}{\cancel{3}}$

$m = -2$

Putting the value of m is equation (i)

$4l - (-2) - 10 = 0$

$4l + 2 - 10 = 0$

$4l - 8 = 0$

$4l = 8$

$l = \frac{2\cancel{8}}{\cancel{4}}$

$l = 2$

Q.8 The expression $lx^3 + mx^2 - 4$ leaves remainder of -3 and 12 when divided by $(x-1)$ and $(x+2)$ respectively. Calculate the value of l and m.

Solution:

$P(x) = lx^3 + mx^2 - 4$

from x-1=0 or x=1

$P(1) = l(1)^3 + m(1)^2 - 4$

$P(1) = l + m - 4$

According to conditions l+m-4=-3

$l + m = 4 - 3$

$l = 1 - m$ ....(i)

From $x + 2 = 0$ or $x = -2$

$P(-2) = l(-2)^3 + m(-2)^2 - 4$

$P(-2) = -8l + 4m - 4$

According to condition

$-8l + 4m - 4 = 12$

Putting the value of l in the equation

$-8[1 - m] + 4m = 16$

$-8 + 8m + 4m = 16$

$12m = 16 + 8$

$12m = 24$

$m = \frac{\cancel{24}^2}{\cancel{12}}$

$m = 2$

Putting the value of m is equation (i)

$l = 1 - 2$

$l = -1$

Result:

$m = 2,\ l = -1$

Q.9 The expression $ax^3 - 9x^2 + bx + 3a$ is exactly divisible by $x^2 - 5x + 6$. Find the value of a and b.

Solution:

Given that

$P(x) = ax^3 - 9x^2 + bx + 3a$

$x^2 - 5x + 6 = x^2 - 2x - 3x + 6$

$= x[x-2] - 3[x-2]$

$= [x-2][x-3]$

Since $x^2 - 5x + 6$ divides $P(x)$ completely then

$(x-2)$ and $(x-3)$ are factors of $P(x)$.

Put $x - 2 = 0 \Rightarrow x = 2$

$P(2)=a(2)^3-9(2)^2+b(2)+3a$

$P(2)=8a-36+2b+3a$

$P(2)=11a+2b-36$

According to condition (x-2) is the factor so

$11a+2b-36=0 \quad \rightarrow (i)$

From $x-3=0$ or $x=3$

$P(3)=a(3)^3-9(3)^2+b(3)+3a$

$P(3)=27a-81+3b+3a$

$P(3)=30a+3b-81$

According to condition (x-3) is the factor so

$30a+3b-81=0 \quad \rightarrow (ii)$

$3(10a+b-27)=0$

$10a+b-27=\frac{0}{3}$

$b=27-10a \quad \rightarrow (iii)$

Putting the value of b in equation (i)

$11a+2[27-10a)-36=0$

$11a+54-20a-36=0$

$-9a+18=0$

$18=9a$

$a=\frac{\cancel{18}\,2}{\cancel{9}}$

$a=2$

Putting the value of a in equation (iii)

$b=27-10(+2)$

$b=27-20$

$b=7$

Result:

$a=2,\ b=7$