Factorization of a Cubic Polynomial

Rational Root Theorem **(K.B)**

Let $a_o x^n + a_1 x^{n-1} + ... + a_{n-1}x + a_n = 0$, $a_o \neq 0$ be a polynomial equation of degree n with integral coefficients. If $\frac{p}{q}$ is a rational root (expressed in lowest terms) of the equation, then p is a factor of the constant term a_n and q is factor of the leading coefficient a_o.

Example **(A.B)**

Factorize the polynomial $x^3 - 4x^2 + x + 6$, by using Factor Theorem

Solution:

We have $P(x) = x^3 - 4x^2 + x + 6$.

Possible factor of the constant term $P = 6$ are $\pm1, \pm2, \pm3$ and ±6 and of leading coefficient $q = 1$ are ±1. Thus the expected zeros (or root) of $P(x) = 0$ are $\frac{p}{q} = \pm1, \pm2,, \pm3$ and ±6 if $x = a$ is a zero of $P(x)$, then $(x - a)$ will be a factor.

We use the hit and trail method to find zeros of $P(x)$ let us try $x = 1$

Now $P(1) = (1)^3 - 4(1)^2 + 1 + 6$

$= 1 - 4 + 1 + 6 = 4 \neq 0$

Hence $x = 1$ is not a zero of $P(x)$.

Again $P(-1) = (-1)^3 - 4(-1)^2 - 1 + 6$

$= -1 - 4 - 1 + 6 = 0$

Hence $x = -1$ is a zero of $P(x)$ and therefore $x - (-1) = (x + 1)$ a factor of $P(x)$

Now

$P(2) = (2)^3 - 4(2)^2 + 2 + 6$

$= 8 - 16 + 2 + 6 = 0$

$\Rightarrow x = 2$ is a root of $P(x)$.

Hence $(x - 2)$ is also a factor of $P(x)$

Similarly $P(3) = (3)^3 - 4(3)^2 + 3 + 6$

$= 27 - 36 + 3 + 6 = 0$

$\Rightarrow x = 3$ is a zero of $P(x)$.

Hence $(x - 3)$ is the third factor of $P(x)$.

Thus the factorized form of

$P(x) = x^3 - 4x^2 + x + 6$

is $P(x) = (x + 1)(x - 2)(x - 3)$

Exercise 5.4

Q.1 $x^3 - 2x^2 - x + 2$

Solution: Given that

$P(x) = x^3 - 2x^2 - x + 2$

P=2 and possible factors of 2 are $\pm1, \pm2$.

Here q=1 and possible factor of 1 are ±1.

So possible factor of P(x) can be $\frac{P}{q} = \pm1, \pm2$

$P(x) = x^3 - 2x^2 - x + 2$

Put x=1

$P(1) = (1)^3 - 2(1)^2 - 1 + 2$

$= 1 - 2 - 1 + 2 = 0$

As remainder is equal to zero, (x-1) is factor.

Put x = - 1

$P(-1) = (-1)^3 - 2(-1)^2 - (-1) + 2$

$= -1 - 2 + 1 + 2 = 0$

As remainder is equal to zero, (x+1) is factor.

Put x=2

$P(2) = (2)^3 - 2(2)^2 - (2) + 2 = 8 - 8 - 2 + 2 = 0$

As remainder is equal to zero, (x-2) is factor.

$x^3 - 2x^2 - x + 2 = (x - 1)(x + 1)(x - 2)$

Q.2 $x^3 - x^2 - 22x + 40$

Solution:

Given that **(K.B)**

$P(x)=x^3-x^2-22x+40$

P = 40 possible factors of 40 are:

$=\pm1,\pm2,\pm4,\pm5,\pm8,\pm10,\pm20,\pm40$

Here q=1 and possible factor of 1are ±1

So possible factor of P(x) will be

$\frac{P}{q}=\pm1,\pm2,\pm4,\pm5,\pm8,\pm10,\pm20,\pm40$

$P(x)=x^3-x^2-22x+40$

Put $x=2$

$P(2)=(2)^3-(2)^2-22(2)+40$

$=8-4-44+40=0$

As remainder is equal to zero, (x-2) is a factor.

Put x=4

$P(4)=(4)^3-(4)^2-22(4)+40$

$=64-16-88+40=0$

As remainder is equal to zero, (x-4) is a factor.

Put x=-5

$P(-5)=(-5)^3-(-5)^2-22(-5)+40$

$=-125-25+110+40$

$=-150+150$

$=0$

As remainder is equal to zero, (x+5) is a factor.

Hence

$x^3-x^2-22x+40=(x-2)(x-4)(x+5)$

Q.3 $x^3-6x^2+3x+10$

Solution:

Given that

$P(x)=x^3-6x^2+3x+10$

P=10

So possible factors of 10 are $\pm1,\pm2,\pm5,\pm10$

Here q=1 So, possible factor of 1are ±1.

So possible of factor of P(x) can be

$\frac{P}{q}=\pm1,\pm2,\pm5,\pm10$

P(x)=x^3-6x^2+3x+10

Put x=-1

$P(-1)=(-1)^3-6(-1)^2+3(-1)+10$

= -1-6-3+10 = 0

As remainder is equal to zero, (x+1) is a factor.

Put x = 2

P(2) = $(2)^3-6(2)^2+3(2)+10$

= 8 − 24+6+10 = 0

As remainder is equal to zero, (x-2) is a factor.

Put x = 5

$P(5)=(5)^3-6(5)^2+3(5)+10=125-150+15+10=0$

As remainder is equal to zero, (x-5) is a factor.

Hence

$x^3-6x^2+3x+10=(x+1)(x-2)(x-5)$

Q.4 $x^3+x^2-10x+8$

Solution:

Given that

P(x) = $x^3+x^2-10x+8$

P=8 So possible factors of 8 are $\pm1,\pm2,\pm4,\pm8$.

Here q=1 So possible factor can be

$\frac{P}{q}=\pm1,\pm2,\pm4,\pm8$

P(x)=x^3+x^2-10x+8

Put x=1

$P(1)=(1)^3+(1)^2-10(1)+8=1+1-10+8=0$

As remainder is equal to zero, (x-1) is a factor.

Put x=2

$P(2)=2^3+2^2-10(2)+8$

$=8+4-20+8$

$=20-20$

$=0$

As remainder is equal to zero, (x-2) is a factor.

Put x= −4

P(−4) = $(-4)^3+(-4)^2-10(-4)+8$

= −64+16+40+8

= −64+64

= 0

As remainder is equal to zero, (x+4) is a factor.

Hence

$x^3+x^2-10x+8=(x-1)(x-2)(x+4)$

Q.5 x^3-2x^2-5x+6

Solution:

Given that

$P(x)=x^3-2x^2-5x+6$

$P=6$ So factors of 6 are $\pm1,\pm2,\pm3,\pm6$

Here q=1, so factors of 1 are ±1.

So possible factors of P(x) can be

$\frac{P}{q}=\pm1,\pm2,\pm3,\pm6$

P(x) = x^3-2x^2-5x+6

Put x=1

P(1) = $(1)^3-2(1)^2-5(1)+6$

$=1-2-5+6$

$=-7+7$

$=0$

Remainder is equal to zero so $(x-1)$ is a factor

Put $x=-2$

$P(-2)=(-2)^3-2(-2)^2-5(-2)+6$

$=-8-8+10+6$

$=-16+16$

$=0$

Remainder is equal to zero so $(x+2)$ is a factor

Put x=3

$P(3)=(3)^3-2(3)^2-5(3)+6$

$=27-6-15+6$

$=27-27$

$=0$

As remainder is equal to zero, $(x-3)$ is a factor.

Hence

$x^3-2x^2-5x+6=(x-1)(x+2)(x-3)$

Q.6 $x^3+5x^2-2x-24$

Solution: Given that

$P(x)=x^3+5x^2-2x-24$

P= -24 So possible factors of 24 are $\pm1,\pm2,\pm3,\pm4,\pm6,\pm8,\pm12,\pm24$

Here q=1. So possible factors of 1 are ±1.

So possible factors of P(x) will be

$\frac{P}{q}=\pm1,\pm2,\pm3,\pm4,\pm6,\pm8,\pm12,\pm24$

$P(x)=x^3+5x^2-2x-24$

Put x = 2

$P(2)=(2)^3+5(2)^2-2(2)-24$

$=8+20-4-24$

$=28-28$

$=0$

As remainder is equal to zero, $(x-2)$ is a factor.

Put $x=-3$

$P(-3)=(-3)^3+5(-3)^2-2(-3)-24$

$=-27+45+6-24$

$=-51+51$

$=0$

Remainder is equal to zero so $(x+3)$ is a factor

Put $x=-4$

$P(-4)=(-4)^3+5(-4)^2-2(-4)-24$

$=-64+80+8-24$

$=-88+88$

$=0$

Remainder is equal to zero so $(x+4)$ is a factor

Hence

$x^3+5x^2-2x-24=(x-2)(x+3)(x+4)$

Q.7 $3x^3-x^2-12x+4$

Solution:

Given that

$P(x)=3x^3-x^2-12x+4$

P=4 So possible factors of 4 are $\pm1,\pm2,\pm4$.

Here q=3 So possible factors of 3 are $\pm1,\pm3$.

So possible factors of P(x) can be

$\frac{P}{q}=\pm1,\pm2,\pm4,\pm\frac{1}{3},\pm\frac{2}{3},\pm\frac{4}{3}$

Put x=2

$P(2)=3(2)^3-(2)^2-12(2)+4$

$=-24-4+24+4$

$=28-28$

$=0$

As remainder is equal to zero, $(x-2)$ is a factor.

Put $x=-2$

$P(-2)=3(-2)^3-(-2)^2-12(-2)+4$

$=-24-4+24+4$

$=28-28$

$= 0$

As remainder is equal to zero, $(x+2)$ is a factor.

Put $x = \frac{1}{3}$

$$P\left(\frac{1}{3}\right) = 3\left(\frac{1}{3}\right)^3 - \left(\frac{1}{3}\right)^2 - 12\left(\frac{1}{3}\right) + 4$$

$$= 3\left(\frac{1}{27}\right) - \frac{1}{9} - 4 + 4$$

$$P\left(\frac{1}{3}\right) = \frac{1}{9} - \frac{1}{9}$$

$= 0$

As remainder is equal to zero, $(3x-1)$ is a factor

Hence

$$3x^3 - x^2 - 10x + 4 = (x-2)(x+2)(3x-1)$$

Q.8 $2x^3 + x^2 - 2x - 1$

Solution:

Given that

$P(x) = 2x^3 + x^2 - 2x - 1$

P= 1, so possible factors of -1 are ± 1.

Here q=2. So possible factors 2 are $\pm 1, \pm 2$.

Therefore, possible values of P(x) will be

$$\frac{P}{q} = \pm 1, \pm 2, \pm\frac{1}{2}$$

$P(x) = 2x^3 + x^2 - 2x - 1$

Put $x=1$

$P(1) = 2(1)^3 + (1)^2 - 2(1) - 1$

$= 2 + 1 - 2 - 1$

$= 3 - 3$

$= 0$

As remainder is equal to zero, $(x-1)$ is a factor.

Put $x = -1$

$$P(-1) = 2(-1)^3 + (-1)^2 - 2(-1) - 1$$

$= 2 + 1 - 2 - 1$

$= 3 - 3$

$= 0$

As remainder is equal to zero, $(x+1)$ is a factor.

Put $x = \frac{-1}{2}$

$$P\left(\frac{-1}{2}\right) = 2\left[\frac{-1}{2}\right]^3 + \left[\frac{-1}{2}\right]^2 - 2\left[\frac{-1}{2}\right] - 1$$

$$P\left(\frac{-1}{2}\right) = 2\left[\frac{-1}{8}\right] + \frac{1}{4} + 1 - 1$$

$$P\left(\frac{-1}{2}\right) = -\frac{1}{4} + \frac{1}{4}$$

$= 0$

$\because x = -\frac{1}{2} \Rightarrow 2x + 1 = 0$

As remainder is equal to zero, $(2x+1)$ is a factor.

Hence $2x^3 + x^2 - 2x - 1 = (x-1)(x+1)(2x+1)$