Mathematics-9
Unit 1 - Overview

Applications of Matrices **(K.B)**

The matrices and determinants are used in the field of mathematics, physics, statistics, Electronics and other branches of science. The matrices have played a very important role in this age of computer science.

The Idea of Matrices **(K.B)**

The idea of matrices was given by Arthur Cayley, an English mathematician of nineteen century who first developed, "Theory of Matrices" in 1858.

Matrix **(K.B)**

(D.G.K 2017, GRW 2017, FSD 2018, SGD 2018)

A rectangular array or a formation of a collection of real number say 0, 1, 2, 3 and 4 and 7 such as $\begin{matrix}1 & 3 & 4\\ 7 & 2 & 0\end{matrix}$ and then enclosed by brackets '[ ]' is said to form a matrix.

The matrices are denoted conventionally by the capital letters A, B, C,…..,M, N etc of the English alphabets.

For example:

$$A=\begin{bmatrix}1 & 3 & 4\\ 7 & 2 & 0\end{bmatrix}, B=\begin{bmatrix}0 & 1\\ 3 & 4\end{bmatrix} \text{ etc.}$$

Rows and Columns of a Matrix **(K.B)**

It is important to understand an entity of a matrix with the following formation.

Rows of a Matrix **(K.B)**

(BWP 2015, 16, SWL 2018)

In matrix, the entries presented in horizontal way are called rows.

$$A=\begin{bmatrix}0 & 1 & 0\\ 1 & 2 & 8\\ 7 & 1 & 5\end{bmatrix}\begin{matrix}\to R_1\\ \to R_2\\ \to R_3\end{matrix}.$$

Columns of a Matrix **(K.B)**

(SGD 2016, 18)

In matrix, all the entries presented in vertical way are called columns of matrix.

Matrix B has three columns as shown by C_1 C_2 and C_3.

$$B=\begin{bmatrix}0 & 1 & 0\\ 1 & 2 & 8\\ 7 & 1 & 5\end{bmatrix}$$
$$\begin{matrix}\downarrow & \downarrow & \downarrow\\ C_1 & C_2 & C_3\end{matrix}$$

Entries or Elements of a Matrix **(K.B)**

The real numbers used in the formation of a matrix are termed as entries or elements of a matrix.

Order of a Matrix **(K.B)**

The number of rows and columns in a matrix specifies its order. If a matrix M has m rows and n columns then M is said to be of order, $m-by-n$.

For example

Order of matrix $\begin{bmatrix}2 & 0 & 5\\ 4 & 1 & 3\end{bmatrix}$ is $2-by-3$

Equal Matrices **(K.B)**

Let A and B be two matrices. Then A is said to be equal to B, and denoted by $A=B,$ if and only if;

(i) the order of A= the order of B

(ii) their corresponding entries are equal.

For example:

$A=\begin{bmatrix}1 & 3\\ -4 & 2\end{bmatrix}$ and $B=\begin{bmatrix}1 & 2+1\\ -4 & 4-2\end{bmatrix}$ are equal matrices.

Types of Matrices

(i) Row Matrix **(K.B)**

A matrix is called a row matrix, if it has only one row.

Example the matrix $M=\begin{bmatrix}2 & -1 & 7\end{bmatrix}$ is a row matrix of order 1–by–2.

(ii) Column Matrix **(K.B)**

A matrix is called a column matrix if it has only one column.

(iii) Rectangular Matrix **(K.B)**

(GRW 2015, MTN 2015, RWP 2016, D.G.K 2018)

A matrix M is called rectangular if, the number of rows of M is not equal to the number of columns of M.

e.g., $B=\begin{bmatrix}a & b & c\\ d & e & d\end{bmatrix}$.

The order of B is 2-by-3

(iv) Square Matrix **(K.B)**

(FSD 2015, 17, LHR 215, SGD 2017)

A matrix is called a square matrix if its number of rows is equal to its number of columns.

e.g., $A=\begin{bmatrix}2 & -1\\ 0 & 3\end{bmatrix}$

the order of A is 2-by-2

(v) Null or Zero Matrix **(K.B)**

(LHR 2018, D.G.K 2015)

A matrix M is called a null or zero matrix if each of its entries is 0.

e.g., $\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$ and $\begin{bmatrix}0\\ 0\end{bmatrix}$

are null matrix of order 2-by-2, 1-by-2, 3-by-3 and 2-by-1 respectively

Note **(U.B)**

Null matrix is represented by O.

(vi) Transpose of a Matrix **(K.B)**

A matrix obtained by changing the rows into columns or columns into rows of a matrix is called transpose of that matrix.

If A is a matrix, then its transpose is denoted by A^t.

e.g., If $A=\begin{bmatrix}1 & 2 & 3\\ 2 & 1 & 0\\ -1 & 4 & -2\end{bmatrix}$,

then $A^t=\begin{bmatrix}1 & 2 & -1\\ 2 & 1 & 4\\ 3 & 0 & -2\end{bmatrix}$

Note **(U.B)**

If a matrix A is of order 2-by-3, then order of its transpose A^t is 3-by-2.

(vii) Negative of a Matrix **(K.B)**

Let A be matrix. Then its negative, $-A$ is obtained by changing the signs of all the entries of A,

i.e., If

$A=\begin{bmatrix}1 & -2\\ 3 & 4\end{bmatrix}$, then $-A=\begin{bmatrix}-1 & 2\\ -3 & -4\end{bmatrix}$

(viii) Symmetric Matrix **(K.B)**

(SGD 2015, 17, BWP 2015, FSD 2016, SWL 2016, 17, MTN 2017)

A square matrix is symmetric if it is equal to its transpose i.e., matrix A is symmetric if $A^t=A$.

e.g.,

(i) If $M=\begin{bmatrix}1 & 2 & 3\\ 2 & -1 & 4\\ 3 & 4 & 0\end{bmatrix}$

is a square matrix, then

$M^t=\begin{bmatrix}1 & 2 & 3\\ 2 & -1 & 4\\ 3 & 4 & 0\end{bmatrix}=M$.

Thus M is a symmetric matrix.

(ix) Skew-Symmetric Matrix **(K.B)**

(D.G.K 2018)

A square matrix A is said to be skew-symmetric if $A^t=-A$.

e.g., $A=\begin{bmatrix}0 & 2 & 3\\ -2 & 0 & 1\\ -3 & -1 & 0\end{bmatrix}$, then

$$A^t=\begin{bmatrix}0 & -2 & -3\\ 2 & 0 & -1\\ 3 & 1 & 0\end{bmatrix}=\begin{bmatrix}0 & -2 & -3\\ -(-2) & 0 & -1\\ -(-3) & -(-1) & 0\end{bmatrix}$$

$$=-\begin{bmatrix}0 & 2 & 3\\ -2 & 0 & 1\\ -3 & -1 & 0\end{bmatrix}=-A$$

Since $A^t=-A$, therefore A is a skew symmetric matrix.

(x) Diagonal Matrix **(K.B)**

(RWP 2015, MTN 2016, BWP 2018)

A square matrix A is called a diagonal matrix if at least any one of the entries of its diagonal is not zero and non-diagonal entries are zero.

i.e., $A=\begin{bmatrix}1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 3\end{bmatrix}$.

Is diagonal matrices of order 3-by-3.

(xi) Scalar Matrix **(K.B)**

(BWP 2015, 18, MTN 2016, FSD 2016, LHR 2017, GRW 2018)

A diagonal matrix is called a scalar matrix, if all the diagonal entries are same and non-zero.

For example $\begin{bmatrix}k & 0 & 0\\ 0 & k & 0\\ 0 & 0 & k\end{bmatrix}$ where k is a constant $\neq 0,1$

For example $A=\begin{bmatrix}2 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 2\end{bmatrix}$

is scalar matrix of order 3-by-3 respectively.

(xii) Identity Matrix **(LHR 2018)** **(K.B)**

A diagonal matrix is called identity (unit) matrix if all diagonal entries are 1 and it is denoted by I.

e.g., $A=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$ is a 3-by-3 identity matrix.

Note **(K.B+U.B)**

(i) The scalar matrix and identity matrix are diagonal matrices.

(ii) Every diagonal matrix is not a scalar or identity matrix.

ADDITION AND SUBTRACTION OF MATRICES

Addition of Matrices **(K.B)**

Let A and B be any two matrices with real number entries. The matrices A and B are conformable for addition, if they have the same order.

e.g., $A=\begin{bmatrix}2 & 3 & 0\\ 1 & 0 & 6\end{bmatrix}$ and $B=\begin{bmatrix}-2 & 3 & 4\\ 1 & 2 & 3\end{bmatrix}$

are conformable for addition.

Addition of A and B, written A+B is obtained by adding the entries of the matrix A to the corresponding entries of the matrix B.

e.g., $A+B=\begin{bmatrix}2 & 3 & 0\\ 1 & 0 & 6\end{bmatrix}+\begin{bmatrix}-2 & 3 & 4\\ 1 & 2 & 3\end{bmatrix}$

$$=\begin{bmatrix}2+(-2) & 3+3 & 0+4\\ 1+1 & 0+2 & 6+3\end{bmatrix}$$

$$=\begin{bmatrix}0 & 6 & 4\\ 2 & 2 & 9\end{bmatrix}$$

Subtraction of Matrices **(K.B)**

If A and B are two matrices of same order, then subtraction of matrix B from matrix A is obtained by subtracting the entries of matrix B from the corresponding entries of matrix A and it is denoted by A – B

e.g.,

$$A=\begin{bmatrix}2 & 3 & 4\\1 & 5 & 0\end{bmatrix} \text{ and } B=\begin{bmatrix}0 & 2 & 2\\-1 & 4 & 3\end{bmatrix} \text{ are}$$

conformable for subtraction.

i.e.,

$$A-B=\begin{bmatrix}2 & 3 & 4\\1 & 5 & 0\end{bmatrix}-\begin{bmatrix}0 & 2 & 2\\-1 & 4 & 3\end{bmatrix}$$

$$=\begin{bmatrix}2-0 & 3-2 & 4-2\\1-(-1) & 5-4 & 0-3\end{bmatrix}=\begin{bmatrix}2 & 1 & 2\\2 & 1 & -3\end{bmatrix}$$

Note **(U.B)**

That the order of a matrix is unchanged under the operation of matrix addition and matrix subtraction.

Multiplication of a Matrix by a Real Number (K.B)

Let A be any matrix and the real number K be a scalar. Then the scalar multiplication of matrix A with K is obtained by multiplying each entry of matrix A with K. It is denoted by KA

$$\text{Let } A=\begin{bmatrix}1 & -1 & 4\\2 & -1 & 0\\-1 & 3 & 2\end{bmatrix}$$

be a matrix of order 3–by–3 and k = –2 be a real number.

Then $kA = (-2)A$

$$=(-2)\begin{bmatrix}1 & -1 & 4\\2 & -1 & 0\\-1 & 3 & 2\end{bmatrix}$$

$$=\begin{bmatrix}(-2)(1) & (-2)(-1) & (-2)(4)\\(-2)(2) & (-2)(-1) & (-2)(0)\\(-2)(-1) & (-2)(3) & (-2)(2)\end{bmatrix}$$

$$kA=\begin{bmatrix}-2 & 2 & -8\\-4 & 2 & 0\\2 & -6 & -4\end{bmatrix}$$

Scalar multiplication of a matrix leaves the order of the matrix unchanged.

Commutative and Associative Laws of Addition of Matrices (K.B)

(a) Commutative law under addition (U.B)

If A and B are two matrices of the same order, then A+B = B+A is called commutative law under addition.

Let

$$A=\begin{bmatrix}2 & 3 & 0\\5 & 6 & 1\\2 & 1 & 3\end{bmatrix}, B=\begin{bmatrix}3 & -2 & 5\\-1 & 4 & 1\\4 & 2 & -4\end{bmatrix}$$

$$\text{Then. } A+B=\begin{bmatrix}2 & 3 & 0\\5 & 6 & 1\\2 & 1 & 3\end{bmatrix}+\begin{bmatrix}3 & -2 & 5\\-1 & 4 & 1\\4 & 2 & -4\end{bmatrix}$$

$$=\begin{bmatrix}2+3 & 3-2 & 0+5\\5-1 & 6+4 & 1+1\\2+4 & 1+2 & 3-4\end{bmatrix}$$

$$A+B=\begin{bmatrix}5 & 1 & 5\\4 & 10 & 2\\6 & 3 & -1\end{bmatrix}$$

$$=\begin{bmatrix}5 & 1 & 5\\4 & 10 & 2\\6 & 3 & -1\end{bmatrix}$$

Thus the commutative law of addition of matrices is verified A + B = B + A

Similarly

$$B+A=\begin{bmatrix}3 & -2 & 5\\-1 & 4 & 1\\4 & 2 & -4\end{bmatrix}+\begin{bmatrix}2 & 3 & 0\\5 & 6 & 1\\2 & 1 & 3\end{bmatrix}$$

(b) Associative Law Under Addition **(U.B)**

If A, B and C are three matrices of same order, then $(A+B)+C=A+(B+C)$ is called associative law under addition.

Let $A=\begin{bmatrix}2 & 3 & 0\\5 & 6 & 1\\2 & 1 & 3\end{bmatrix}, B=\begin{bmatrix}3 & -2 & 5\\-1 & 4 & 1\\4 & 2 & -4\end{bmatrix}$

and $C=\begin{bmatrix}1 & 2 & 3\\-2 & 0 & 4\\1 & 2 & 0\end{bmatrix}$

Then $(A+B)+C=\left(\begin{bmatrix}2 & 3 & 0\\5 & 6 & 1\\2 & 1 & 3\end{bmatrix}+\begin{bmatrix}3 & -2 & 5\\-1 & 4 & 1\\4 & 2 & -4\end{bmatrix}\right)+\begin{bmatrix}1 & 2 & 3\\-2 & 0 & 4\\1 & 2 & 0\end{bmatrix}$

$$=\begin{bmatrix}2+3 & 3-2 & 0+5\\5-1 & 6+4 & 1+1\\2+4 & 1+2 & 3-4\end{bmatrix}+\begin{bmatrix}1 & 2 & 3\\-2 & 0 & 4\\1 & 2 & 0\end{bmatrix}$$

$$=\begin{bmatrix}5 & 1 & 5\\4 & 10 & 2\\6 & 3 & -1\end{bmatrix}+\begin{bmatrix}1 & 2 & 3\\-2 & 0 & 4\\1 & 2 & 0\end{bmatrix}=\begin{bmatrix}6 & 3 & 8\\2 & 10 & 6\\7 & 5 & -1\end{bmatrix}$$

$$A+(B+C)=\begin{bmatrix}2 & 3 & 0\\5 & 6 & 1\\2 & 1 & 3\end{bmatrix}+\left(\begin{bmatrix}3 & -2 & 5\\-1 & 4 & 1\\4 & 2 & -4\end{bmatrix}+\begin{bmatrix}1 & 2 & 3\\-2 & 0 & 4\\1 & 2 & 0\end{bmatrix}\right)$$

$$=\begin{bmatrix}2 & 3 & 0\\5 & 6 & 1\\2 & 1 & 3\end{bmatrix}+\begin{bmatrix}3+1 & -2+2 & 5+3\\-1-2 & 4+0 & 1+4\\4+1 & 2+2 & -4+0\end{bmatrix}$$

$$=\begin{bmatrix}2 & 3 & 0\\5 & 6 & 1\\2 & 1 & 3\end{bmatrix}+\begin{bmatrix}4 & 0 & 8\\-3 & 4 & 5\\5 & 4 & -4\end{bmatrix}=\begin{bmatrix}6 & 3 & 8\\2 & 10 & 6\\7 & 5 & -1\end{bmatrix}$$

Thus the associative law of addition is verified:

$(A+B)+C=A+(B+C)$

Additive Identity of Matrices (U.B)

If A and B are two matrices of same order and A+B = A = B+A

then matrix B is called additive identity of matrix A

For any matrix A and zero matrix O of same order, O is called additive identity of A as $A+0=A=0+A$

e.g., Let $A=\begin{bmatrix}1 & 2\\ 3 & 5\end{bmatrix}$ and $O=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$

then $A+O=\begin{bmatrix}1 & 2\\ 3 & 5\end{bmatrix}+\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}=\begin{bmatrix}1 & 2\\ 3 & 5\end{bmatrix}=A$

$O+A=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}+\begin{bmatrix}1 & 2\\ 3 & 5\end{bmatrix}=\begin{bmatrix}1 & 2\\ 3 & 5\end{bmatrix}=A$

Additive Inverse of a Matrix (U.B)

If A and B are two matrices of same order such that A+B = O = B+A

Then A and B are called additive inverse of each other.

Additive inverse of any matrix A is obtained by changing to negative of the symbols (entries) of each non zero entry of A

Let $A=\begin{bmatrix}1 & 2 & 1\\ 0 & -1 & -2\\ 3 & 1 & 0\end{bmatrix}$

then

$$B=(-A)=-\begin{bmatrix}1 & 2 & 1\\ 0 & -1 & -2\\ 3 & 1 & 0\end{bmatrix}=\begin{bmatrix}-1 & -2 & -1\\ 0 & 1 & 2\\ -3 & -1 & 0\end{bmatrix}$$

is additive inverse of A.

It can be verified as

$$A+B=\begin{bmatrix}1 & 2 & 1\\ 0 & -1 & -2\\ 3 & 1 & 0\end{bmatrix}+\begin{bmatrix}-1 & -2 & -1\\ 0 & 1 & 2\\ -3 & -1 & 0\end{bmatrix}$$

$$=\begin{bmatrix}(1)+(-1) & (2)+(-2) & (1)+(-1)\\ 0+0 & (-1)+(1) & (-2)+(2)\\ (3)+(-3) & (1)+(-1) & 0+0\end{bmatrix}$$

$$=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}=O$$

$$B+A=\begin{bmatrix}-1 & -2 & -1\\ 0 & 1 & 2\\ -3 & -1 & 0\end{bmatrix}+\begin{bmatrix}1 & 2 & 1\\ 0 & -1 & -2\\ 3 & 1 & 0\end{bmatrix}$$

$$=\begin{bmatrix}(-1)+(1) & (-2)+(2) & (-1)+(1)\\ 0+0 & (1)+(-1) & (2)+(-2)\\ (-3)+(3) & (-1)+(1) & 0+0\end{bmatrix}$$

$$=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}=O$$

Since A+B = 0 = B+A

Therefore, A and B are additive inverse of each other.

Multiplication of Matrices

Two matrices A and B are conformable for multiplication giving product AB if the number if columns of A is equal to the number of rows of B.

e.g., Let $A=\begin{bmatrix}1 & 2\\ 3 & 0\end{bmatrix}$ and $B=\begin{bmatrix}4\\ 1\end{bmatrix}$.

Here number of columns of A is equal to the number of rows of B. So A and B matrices are conformable for multiplication.

Multiplication of two matrices is explained by the following examples.

(i) If $A=\begin{bmatrix}1 & 2\end{bmatrix}$ and $B=\begin{bmatrix}2 & 0\\ 3 & 1\end{bmatrix}$

Then $AB=\begin{bmatrix}1 & 2\end{bmatrix}\begin{bmatrix}2 & 0\\ 3 & 1\end{bmatrix}$

$=\begin{bmatrix}1\times2+2\times3 & 1\times0+2\times1\end{bmatrix}$

$=\begin{bmatrix}2+6 & 0+2\end{bmatrix}=\begin{bmatrix}8 & 2\end{bmatrix}$

is a 1-by-2 matrix

(ii) If $A=\begin{bmatrix}1 & 3\\ 2 & -3\end{bmatrix}$ $B=\begin{bmatrix}-1 & 0\\ 3 & 2\end{bmatrix}$

Then $AB=\begin{bmatrix}1 & 3\\ 2 & -3\end{bmatrix}\times\begin{bmatrix}-1 & 0\\ 3 & 2\end{bmatrix}$

$$=\begin{bmatrix}1\times(-1)+3\times3 & 1\times0+3\times2\\ 2(-1)+(-3)(3) & 2\times0+(-3)(2)\end{bmatrix}$$

$$AB=\begin{bmatrix}-1+9 & 0+6\\ -2-9 & 0-6\end{bmatrix}=\begin{bmatrix}8 & 6\\ -11 & -6\end{bmatrix}$$

Associative Law Under Multiplication

If A, B and C are three matrices conformable for multiplication then associative law under multiplication is given as

$(AB)C=A(BC)$ **(U.B)**

e.g., If

$$A=\begin{bmatrix}2 & 3\\ -1 & 0\end{bmatrix},\ B=\begin{bmatrix}0 & 1\\ 3 & 1\end{bmatrix}\text{ and } C=\begin{bmatrix}2 & 2\\ -1 & 0\end{bmatrix}$$

Then

L.H.S. = (AB)C

$$(AB)C=\left(\begin{bmatrix}2 & 3\\ -1 & 0\end{bmatrix}\begin{bmatrix}0 & 1\\ 3 & 1\end{bmatrix}\right)\begin{bmatrix}2 & 2\\ -1 & 0\end{bmatrix}$$

$$=\begin{bmatrix}2\times0+3\times3 & 2\times1+3\times1\\ -1\times0+0\times3 & -1\times1+0\times1\end{bmatrix}\begin{bmatrix}2 & 2\\ -1 & 0\end{bmatrix}$$

$$=\begin{bmatrix}0+9 & 2+3\\ 0+0 & -1+0\end{bmatrix}\begin{bmatrix}2 & 2\\ -1 & 0\end{bmatrix}$$

$$=\begin{bmatrix}9 & 5\\ 0 & -1\end{bmatrix}\begin{bmatrix}2 & 2\\ -1 & 0\end{bmatrix}$$

$$=\begin{bmatrix}9\times2+5\times(-1) & 9\times2+5\times0\\ 0\times2+(-1)\times(-1) & 0\times2+(-1)\times(0)\end{bmatrix}$$

$$=\begin{bmatrix}18-5 & 18+0\\ 0+1 & 0+0\end{bmatrix}=\begin{bmatrix}13 & 18\\ 1 & 0\end{bmatrix}$$

R.H.S. =

$$A(BC)=\begin{bmatrix}2 & 3\\ -1 & 0\end{bmatrix}\left(\begin{bmatrix}0 & 1\\ 3 & 1\end{bmatrix}\begin{bmatrix}2 & 2\\ -1 & 0\end{bmatrix}\right)$$

$$=\begin{bmatrix}2 & 3\\ -1 & 0\end{bmatrix}\begin{bmatrix}0\times2+1\times(-1) & 0\times2+1\times0\\ 3\times2+1\times-1 & 3\times2+1\times0\end{bmatrix}$$

$$=\begin{bmatrix}2 & 3\\ -1 & 0\end{bmatrix}\begin{bmatrix}-1 & 0\\ 5 & 6\end{bmatrix}$$

$$=\begin{bmatrix}2(-1)+3\times5 & 2\times0+3\times6\\ (-1)(-1)+0\times5 & -1\times0+0\times6\end{bmatrix}$$

$$=\begin{bmatrix}-2+15 & 0+18\\ 1+0 & 0+0\end{bmatrix}$$

$$=\begin{bmatrix}13 & 18\\ 1 & 0\end{bmatrix}=(AB)C$$

The associative law under multiplication of matrices is verified.

Distributive Laws of Multiplication over Addition and Subtraction

(a) Let A, B and C be three matrices. Then distributive laws of multiplication over addition are given below.

(i) $A(B+C)=AB+AC$ (Left distributive law) **(U.B)**

(ii) $(A+B)C=AC+BC$ (Right distributive law)

$$\text{Let } A=\begin{bmatrix}2 & 3\\ -1 & 0\end{bmatrix},B=\begin{bmatrix}0 & 1\\ 3 & 1\end{bmatrix}\text{ and } C=\begin{bmatrix}2 & 2\\ -1 & 0\end{bmatrix},$$

Then

L.H.S. = A (B+C)… (i)

$$=\begin{bmatrix}2 & 3\\ -1 & 0\end{bmatrix}\left(\begin{bmatrix}0 & 1\\ 3 & 1\end{bmatrix}+\begin{bmatrix}2 & 2\\ -1 & 0\end{bmatrix}\right)$$

$$=\begin{bmatrix}2 & 3\\ -1 & 0\end{bmatrix}\begin{bmatrix}0+2 & 1+2\\ 3-1 & 1+0\end{bmatrix}$$

$$=\begin{bmatrix}2 & 3\\ -1 & 0\end{bmatrix}\begin{bmatrix}2 & 3\\ 2 & 1\end{bmatrix}=\begin{bmatrix}2\times2+3\times2 & 2\times3+3\times1\\ -1\times2+0\times2 & -1\times3+0\times1\end{bmatrix}$$

$$=\begin{bmatrix}4+6 & 6+3\\ -2+0 & -3+0\end{bmatrix}=\begin{bmatrix}10 & 9\\ -2 & -3\end{bmatrix}$$

R.H.S. = AB + AC

$$=\begin{bmatrix}2 & 3\\ -1 & 0\end{bmatrix}\begin{bmatrix}0 & 1\\ 3 & 1\end{bmatrix}+\begin{bmatrix}2 & 3\\ -1 & 0\end{bmatrix}\begin{bmatrix}2 & 2\\ -1 & 0\end{bmatrix}$$

$$=\begin{bmatrix}2\times0+3\times3 & 2\times1+3\times1\\ -1\times0+0\times3 & -1\times1+0\times1\end{bmatrix}+\begin{bmatrix}2\times2+3(-1) & 2\times2+3\times0\\ -1\times2+0\times(-1) & -1\times2+0\times0\end{bmatrix}$$

$$=\begin{bmatrix}9 & 5\\ 0 & -1\end{bmatrix}+\begin{bmatrix}1 & 4\\ -2 & -2\end{bmatrix}$$

$$=\begin{bmatrix}9+1 & 5+4\\ 0-2 & -1-2\end{bmatrix}=\begin{bmatrix}10 & 9\\ -2 & -3\end{bmatrix}=\text{L.H.S.}$$

Which shows that

$A(B+C)=AB+AC$ similarly we can verify…(ii)

(b) Similarly the distributive laws of multiplication over subtraction are as follow.

(i) $A(B-C)=AB-AC$

(ii) $(A-B)C=AC-BC$

Let

$A=\begin{bmatrix}2 & 3\\1 & 0\end{bmatrix}, B=\begin{bmatrix}-1 & 1\\1 & 0\end{bmatrix} \text{and } C=\begin{bmatrix}2 & 1\\1 & 2\end{bmatrix}$, then in… (i)

L.H.S. = A (B–C)… (i)

$$=\begin{bmatrix}2 & 3\\0 & 1\end{bmatrix}\left(\begin{bmatrix}-1 & 1\\1 & 0\end{bmatrix}-\begin{bmatrix}2 & 1\\1 & 2\end{bmatrix}\right)$$

$$=\begin{bmatrix}2 & 3\\0 & 1\end{bmatrix}\left(\begin{bmatrix}-1-2 & 1-1\\1-1 & 0-2\end{bmatrix}\right)=\begin{bmatrix}2 & 3\\0 & 1\end{bmatrix}\begin{bmatrix}-3 & 0\\0 & -2\end{bmatrix}$$

$$=\begin{bmatrix}(2)(-3)+(3)(0) & 2\times0+3(-2)\\(0)(-3)+1\times0 & 0\times0+(1)(-2)\end{bmatrix}$$

$$=\begin{bmatrix}-6+0 & 0-6\\0+0 & 0-2\end{bmatrix}=\begin{bmatrix}-6 & -6\\0 & -2\end{bmatrix}$$

R.H.S.= $AB-AC$

$$=\begin{bmatrix}2 & 3\\0 & 1\end{bmatrix}\begin{bmatrix}-1 & 1\\1 & 0\end{bmatrix}-\begin{bmatrix}2 & 3\\0 & 1\end{bmatrix}\begin{bmatrix}2 & 1\\1 & 2\end{bmatrix}$$

$$=\begin{bmatrix}(2)(-1)+(3)(1) & 2(1)+3(0)\\(0)(-1)+1(1) & 0(1)+1(0)\end{bmatrix}$$

$$-\begin{bmatrix}2\times2+3\times1 & 2\times1+3\times2\\0\times2+1\times1 & 0\times1+1\times2\end{bmatrix}$$

$$=\begin{bmatrix}1 & 2\\1 & 0\end{bmatrix}-\begin{bmatrix}7 & 8\\1 & 2\end{bmatrix}=\begin{bmatrix}1-7 & 2-8\\1-1 & 0-2\end{bmatrix}=\begin{bmatrix}-6 & -6\\0 & -2\end{bmatrix}$$

Which shows that

$A(B-C)=AB-AC$

Commutative Law of Multiplication of Matrices **(U.B)**

Consider the matrices

$A=\begin{bmatrix}0 & 1\\2 & 3\end{bmatrix}$ and $B=\begin{bmatrix}1 & 0\\0 & -2\end{bmatrix}$ then

$$AB=\begin{bmatrix}0 & 1\\2 & 3\end{bmatrix}\begin{bmatrix}1 & 0\\0 & -2\end{bmatrix}$$

$$=\begin{bmatrix}0\times1+1\times0 & 0\times0+1\times(-2)\\2\times1+3\times0 & 2\times0+3(-2)\end{bmatrix}$$

$$=\begin{bmatrix}0 & -2\\2 & -6\end{bmatrix}$$

And

$$BA=\begin{bmatrix}1 & 0\\0 & -2\end{bmatrix}\begin{bmatrix}0 & 1\\2 & 3\end{bmatrix}$$

$$=\begin{bmatrix}1\times1+0\times2 & 1\times1+0\times3\\0\times0+(-2)\times2 & 0\times1+3(-2)\end{bmatrix}$$

$$=\begin{bmatrix}0 & 1\\-4 & -6\end{bmatrix}$$

Which shows that, $AB\neq BA$

Commutative law under multiplication in matrices does not hold in general i.e., if A and B are two matrices, then $AB\neq BA$.

Commutative law under multiplication holds in particular case.

e.g., If $A=\begin{bmatrix}2 & 0\\0 & 1\end{bmatrix}$ and $B=\begin{bmatrix}-3 & 0\\0 & 4\end{bmatrix}$ then,

$$AB=\begin{bmatrix}2 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}-3 & 0\\0 & 4\end{bmatrix}$$

$$=\begin{bmatrix}2\times(-3)+0\times0 & 2\times0+0\times4\\0\times(-3)+1\times0 & 0\times0+1\times4\end{bmatrix}$$

$$=\begin{bmatrix}-6 & 0\\0 & 4\end{bmatrix}$$

And

$$BA=\begin{bmatrix}-3 & 0\\0 & 4\end{bmatrix}\begin{bmatrix}2 & 0\\0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}-3\times2+0\times0 & -3\times0+0\times1\\0\times2+4\times0 & 0\times0+4\times1\end{bmatrix}$$

$$=\begin{bmatrix}-6 & 0\\0 & 4\end{bmatrix}$$

which shows that *AB* = *BA*.

Multiplicative Identity of Matrices

(U.B)

Let A be a matrix, another matrix B is called the identity matrix of A under multiplication if $AB = A = BA$

If $A=\begin{bmatrix}1 & 0\\0 & -3\end{bmatrix}$, $B=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$, then

$$AB=\begin{bmatrix}1 & 2\\0 & -3\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}1\times1+2\times0 & 1\times0+2\times1\\0\times1+(-3)\times0 & 0\times0+(-3)(1)\end{bmatrix}$$

$$=\begin{bmatrix}1 & 2\\0 & -3\end{bmatrix}$$

$$BA=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}1 & 2\\0 & -3\end{bmatrix}$$

$$=\begin{bmatrix}1\times1+0\times0 & 1\times2+0\times(-3)\\0\times1+1\times0 & 0\times2+1\times(-3)\end{bmatrix}$$

$$=\begin{bmatrix}1 & 2\\0 & -3\end{bmatrix}$$

which shows that $AB = A = BA$.

Verification of $(AB)^t = B^tA^t$:

Let, $A=\begin{bmatrix}2 & 1\\0 & -1\end{bmatrix}$ and $B=\begin{bmatrix}1 & 3\\-2 & 0\end{bmatrix}$

L.H.S. = $(AB)^t$

$$=\left(\begin{bmatrix}2 & 1\\0 & -1\end{bmatrix}\begin{bmatrix}1 & 3\\-2 & 0\end{bmatrix}\right)^t$$

$$=\begin{bmatrix}2\times1+1\times(-2) & 2\times3+1\times0\\0\times1+(-1)\times(-2) & 0\times3+(-1)\times0\end{bmatrix}^t$$

$$=\begin{bmatrix}2-2 & 6+0\\0+2 & 0+0\end{bmatrix}^t=\begin{bmatrix}0 & 6\\2 & 0\end{bmatrix}^t$$

$$=\begin{bmatrix}0 & 2\\6 & 0\end{bmatrix}$$

R.H.S. = B^tA^t

$$(A)^t=\begin{bmatrix}2 & 1\\0 & -1\end{bmatrix}^t=\begin{bmatrix}2 & 0\\1 & -1\end{bmatrix}$$ and

$$(B)^t=\begin{bmatrix}1 & 3\\-2 & 0\end{bmatrix}^t=\begin{bmatrix}1 & -2\\3 & 0\end{bmatrix}$$

$$B^tA^t=\begin{bmatrix}1 & -2\\3 & 0\end{bmatrix}\begin{bmatrix}2 & 0\\1 & -1\end{bmatrix}$$

$$=\begin{bmatrix}1\times2+(-2)\times1 & 1\times0+(-2)(-1)\\3\times2+0\times1 & 3\times0+0\times(-1)\end{bmatrix}$$

$$=\begin{bmatrix}2-2 & 0+2\\6+0 & 0+0\end{bmatrix}$$

$$B^tA^t=\begin{bmatrix}0 & 2\\6 & 0\end{bmatrix}=\text{L.H.S.}.$$

Thus $(AB)^t=B^tA^t$

Multiplicative Inverse of a Matrix

Determinant of a 2-by-2 Matrix (K.B)

Let $A=\begin{bmatrix}a & b\\c & d\end{bmatrix}$ be a 2-by-2 square matrix.

The determinant of A, denoted by A or $|A|$ is defined as

$$|A|=\det\begin{bmatrix}a & b\\c & d\end{bmatrix}=\begin{vmatrix}a & b\\c & d\end{vmatrix}=ad-bc=\lambda\in R$$

e.g., Let $B=\begin{bmatrix}1 & 1\\-2 & 3\end{bmatrix}$.

Then

$$|B|=\det=B\begin{vmatrix}1 & 1\\-2 & 3\end{vmatrix}=1\times3-(-2)(1)=3+2=5$$

If $M=\begin{bmatrix}2 & 6\\1 & 3\end{bmatrix}$, then

$$\det M=\begin{vmatrix}2 & 6\\1 & 3\end{vmatrix}=2\times3-1\times6=0$$

Singular Matrix (K.B)

(GRW 2017, SWL 2016, MTN 2016, 17 FSD 2018)

A square matrix A is called singular if the determinant of A is equal to zero.

For example, $A=\begin{bmatrix}1 & 2\\0 & 0\end{bmatrix}$ is a singular matrix, since $\det A=1\times0-0\times2=0$.

Non-Singular Matrix (K.B)

(GRW 2017, SWL 2016, MTN 2016, 17 FSD 2018)

A square matrix A is called non-singular if the determinant of A is not equal to zero.

For example $A=\begin{bmatrix}1 & 1\\0 & 2\end{bmatrix}$ is non-singular,

Since $\det A=1\times 2-0\times 1=2\neq 0$.

Adjoint of a Matrix **(K.B)**

Adjoint of a square matrix $A=\begin{bmatrix}a & b\\c & d\end{bmatrix}$ is obtained by interchanging the diagonal entries and changing the sign of other entires. Adjoint of matrix A is denoted as Adj A.

i.e., $\text{Adj A}=\begin{bmatrix}d & -b\\-c & a\end{bmatrix}$

Multiplicative Inverse of Non Singular Matrix **(K.B)**

Let A and B be two non-singular square matrices of same order. Then A and B are said to be multiplicative inverse of each other if

$AB=BA=I$

The inverse of A is denoted by A^{-1}, thus

$AA^{-1}=A^{-1}A=I$

Inverse of a matrix is possible only if matrix is non singular.

Note

Inverse of identity matrix is identity matrix

Inverse of a Matrix using Adjoint **(K.B)**

Let $M=\begin{bmatrix}a & b\\c & d\end{bmatrix}$ be a square matrix. To find the inverse of M, i.e, M^{-1} first we find the determinant as inverse is possible only of a non-singular matrix.

$$|M|=\begin{vmatrix}a & b\\c & d\end{vmatrix}=ad-bc\neq 0$$

and $\text{Adj}M=\begin{bmatrix}d & -b\\-c & a\end{bmatrix}$, then $M^{-1}=\dfrac{Adj\,M}{|M|}$

$$M^{-1}=\frac{\begin{bmatrix}d & -b\\-c & a\end{bmatrix}}{ad-bc}$$

e.g., Let $A=\begin{bmatrix}2 & 1\\-1 & -3\end{bmatrix}$, then

$$|A|=\begin{vmatrix}2 & 1\\-1 & -3\end{vmatrix}=-6-(-1)=-6+1=-5\neq 0$$

thus $A^{-1}=\dfrac{Adj\ A}{|A|}=\dfrac{\begin{bmatrix}-3 & -1\\1 & 2\end{bmatrix}}{-5}$

$$=\frac{-1}{5}\begin{bmatrix}-3 & -1\\1 & 2\end{bmatrix}=\begin{bmatrix}\frac{3}{5} & \frac{1}{5}\\-\frac{1}{5} & -\frac{2}{5}\end{bmatrix}$$

and $AA^{-1}=\begin{bmatrix}-2 & 1\\-1 & -3\end{bmatrix}\begin{bmatrix}\frac{3}{5} & \frac{1}{5}\\-\frac{1}{5} & -\frac{2}{5}\end{bmatrix}$

$$=\begin{bmatrix}\frac{6}{5}-\frac{1}{5} & \frac{2}{5}-\frac{2}{5}\\\frac{-3}{5}+\frac{3}{5} & \frac{-1}{5}+\frac{6}{5}\end{bmatrix}$$

$$AA^{-1}=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}=I=A^{-1}A$$

Verification of $(AB)^{-1}=B^{-1}A^{-1}$: (U.B)

Let $A=\begin{bmatrix}3 & 1\\-1 & 0\end{bmatrix}$ and $B=\begin{bmatrix}0 & -1\\3 & 2\end{bmatrix}$

Then $\det A=3\times 0-(-1)\times 1=1\neq 0$

And det. $B=0\times 2-3(-1)=3\neq 0$

Therefore A and B are invertible i.e. their inverses exist. Then, to verify the law of inverse of the product, take

$$AB=\begin{bmatrix}3 & 1\\-1 & 0\end{bmatrix}\begin{bmatrix}0 & -1\\3 & 2\end{bmatrix}$$

$$AB=\begin{bmatrix}3\times 0+1\times 3 & 3\times(-1)+1\times 2\\-1\times 0+0\times 3 & -1\times(-1)+0\times 2\end{bmatrix}$$

$$AB=\begin{bmatrix}3 & -1\\0 & 1\end{bmatrix}$$

$$\Rightarrow \det(AB)=\begin{vmatrix}3 & -1\\0 & 1\end{vmatrix}=3\neq 0$$

and L.H.S.$=(AB)^{-1}=\frac{1}{3}\begin{bmatrix}1 & 1\\0 & 3\end{bmatrix}=\begin{bmatrix}1/3 & 1/3\\0 & 1\end{bmatrix}$

R.H.S.$=B^{-1}A^{-1}$ where $B^{-1}=\frac{1}{3}\begin{bmatrix}2 & 1\\-3 & 0\end{bmatrix}$,

$A^{-1}=\frac{1}{1}\begin{bmatrix}0 & -1\\1 & 3\end{bmatrix}$

$=\frac{1}{3}\begin{bmatrix}2 & 1\\-3 & 0\end{bmatrix}\cdot\frac{1}{1}\begin{bmatrix}0 & -1\\1 & 3\end{bmatrix}$

$=\frac{1}{3}\begin{bmatrix}2\times0+1\times1 & 2\times(-1)+1\times3\\-3\times0+0\times1 & -3\times(-1)+0\times3\end{bmatrix}$

$=\frac{1}{3}\begin{bmatrix}0+1 & -2+3\\0 & 3\end{bmatrix}=\frac{1}{3}\begin{bmatrix}1 & 1\\0 & 3\end{bmatrix}$

$=\begin{bmatrix}1/3 & 1/3\\0 & 1\end{bmatrix}=(AB)^{-1}$

Thus the law $(AB)^{-1}=B^{-1}A^{-1}$ is verified

Solution of Simultaneous Linear Equations (K.B)

System of two linear equations in two variables in general form is given as:

$ax + by = m$

$cx + dy = n$

where a, b, c, d, m and n are real numbers.

This system is also called simultaneous linear equations.

(i) Solution of System of Linear Equation by Matrix Inversion Method (U.B)

Consider the following system of linear equations.

$ax + by = m$

$cx + dy = n$

Writing in matrices form

$$\begin{bmatrix}a & b\\c & d\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}m\\n\end{bmatrix}$$

Let $AX = B$

where $A=\begin{bmatrix}a & b\\c & d\end{bmatrix}$, $X=\begin{bmatrix}x\\y\end{bmatrix}$ and $B=\begin{bmatrix}m\\n\end{bmatrix}$

or $X = A^{-1}B$

or $X=\frac{Adj\ A}{|A|}\times B \rightarrow (i)$

$\because\ A^{-1}=\frac{Adj\ A}{|A|}$ and $|A|\neq 0$

Here, $|A| = ad - bc \neq 0$

Equation (i) $\Rightarrow$

$$\begin{bmatrix}x\\y\end{bmatrix}=\frac{\begin{bmatrix}d & -b\\-c & a\end{bmatrix}\begin{bmatrix}m\\n\end{bmatrix}}{ad-bc}$$

$$=\begin{bmatrix}\frac{dm-bm}{ad-bc}\\ \frac{-cm+an}{ad-bc}\end{bmatrix}$$

$\Rightarrow x=\frac{dm-bn}{ad-bc}$ and $y=\frac{an-cm}{ad-bc}$

(ii) Cramer's Rule (K.B)

Consider the following system of linear equations.

$ax + by = m$

$cx + by = n$

We know that

$AX = B$

where $A=\begin{bmatrix}a & b\\c & d\end{bmatrix}$, $X=\begin{bmatrix}x\\y\end{bmatrix}$ and $B=\begin{bmatrix}m\\n\end{bmatrix}$

Or $X = A^{-1}B$ or $X=\frac{Adj\ A}{|A|}\times B$

Or $\begin{bmatrix}x\\y\end{bmatrix}=\frac{\begin{bmatrix}d & -b\\-c & a\end{bmatrix}\begin{bmatrix}m\\n\end{bmatrix}}{|A|}=\frac{\begin{bmatrix}dm-bn\\-cm+an\end{bmatrix}}{|A|}$

$$=\begin{bmatrix}\frac{dm-bn}{|A|}\\ \frac{-cm+an}{|A|}\end{bmatrix}$$

Or $x=\frac{dm-bn}{|A|}=\frac{|A_x|}{|A|}$

and $y=\frac{an-cm}{|A|}=\frac{|A_y|}{|A|}$

where $|A_x| = \begin{vmatrix} m & b \\ n & d \end{vmatrix}$ and $|A_y| = \begin{vmatrix} a & m \\ c & n \end{vmatrix}$

Example # 1 **(K.B)**

Solve the following system by using matrix inversion method

$4x - 2y = 8$

$3x + y = -4$

Solution:

$4x - 2y = 8$

$3x + y = -4$

Writing in matrix form

$$\begin{bmatrix} 4 & -2 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ -4 \end{bmatrix}$$

Let $AX = B$

Or

$$X = A^{-1}B \quad \text{or} \quad X = \frac{Adj\,A}{|A|} \times B \rightarrow (i)$$

Here

$$A = \begin{bmatrix} 4 & -2 \\ 3 & 1 \end{bmatrix}$$

$$A = 4\times1 - 3(-2) = 4 + 6 = 10 \neq 0$$

So A^{-1} is possible.

$$Adj\,A = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$$

Putting the values in equation (i)

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} 8 \\ -4 \end{bmatrix}$$

$$= \frac{1}{10}\begin{bmatrix} 8-8 \\ -24-16 \end{bmatrix}$$

$$= \frac{1}{10}\begin{bmatrix} 0 \\ -40 \end{bmatrix} = \begin{bmatrix} 0 \\ -4 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ -4 \end{bmatrix}$$

By comparing, we get

$\Rightarrow x = 0, \quad y = -4$

$\therefore \text{Solution Set} = \{(0, -4)\}$

Example # 2: **(A.B)**

Solve the following system of linear equations by using Cramer's rule

$3x - 2y = 1$

$-2x + 3y = 2$

Solution:

$3x - 2y = 1$

$-2x + 3y = 2$

Writing in matrix form

$$\begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Here

$$A = \begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix}, A_x = \begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix}, A_y = \begin{bmatrix} 3 & 1 \\ -2 & 2 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 3 & -2 \\ -2 & 3 \end{vmatrix} = 9 - 4 = 5 \neq 0$$

(A is non singular)

$$x = \frac{|A_x|}{|A|} = \frac{\begin{vmatrix} 1 & -2 \\ 2 & 3 \end{vmatrix}}{5} = \frac{3+4}{5} = \frac{7}{5}$$

$$y = \frac{|A_y|}{|A|} - \frac{\begin{vmatrix} 3 & 1 \\ -2 & 2 \end{vmatrix}}{5} = \frac{6+2}{5} = \frac{8}{5}$$

$$\therefore \text{Solution Set} = \left\{\left(\frac{7}{5}, \frac{8}{5}\right)\right\}$$