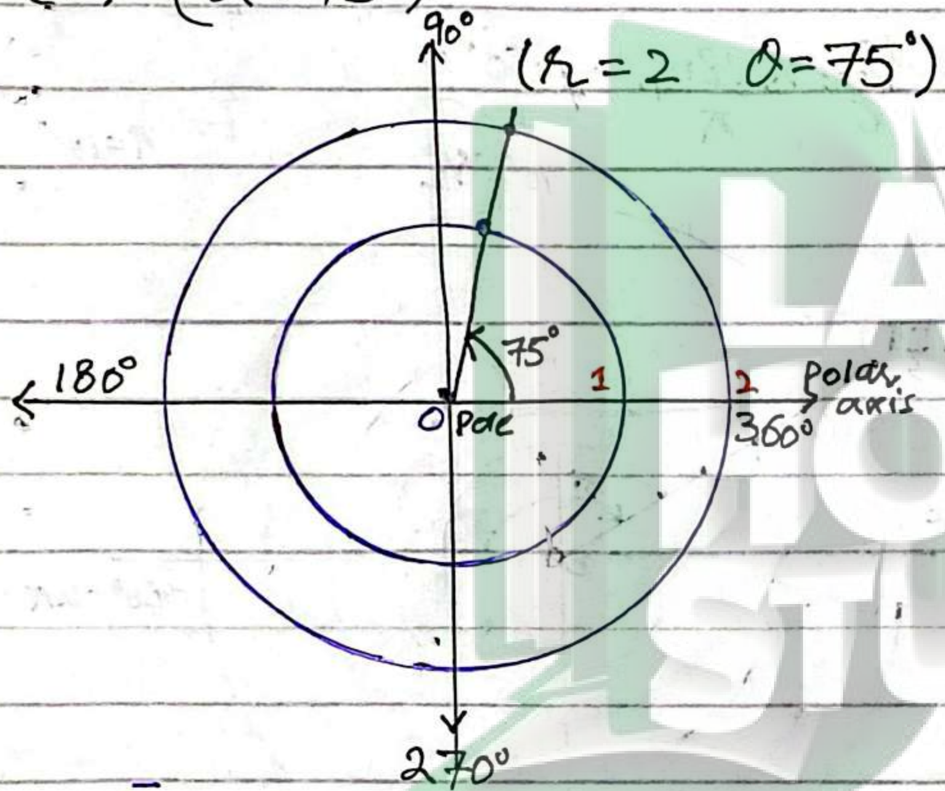
Exercise # 1.5 (Complex Numbers)

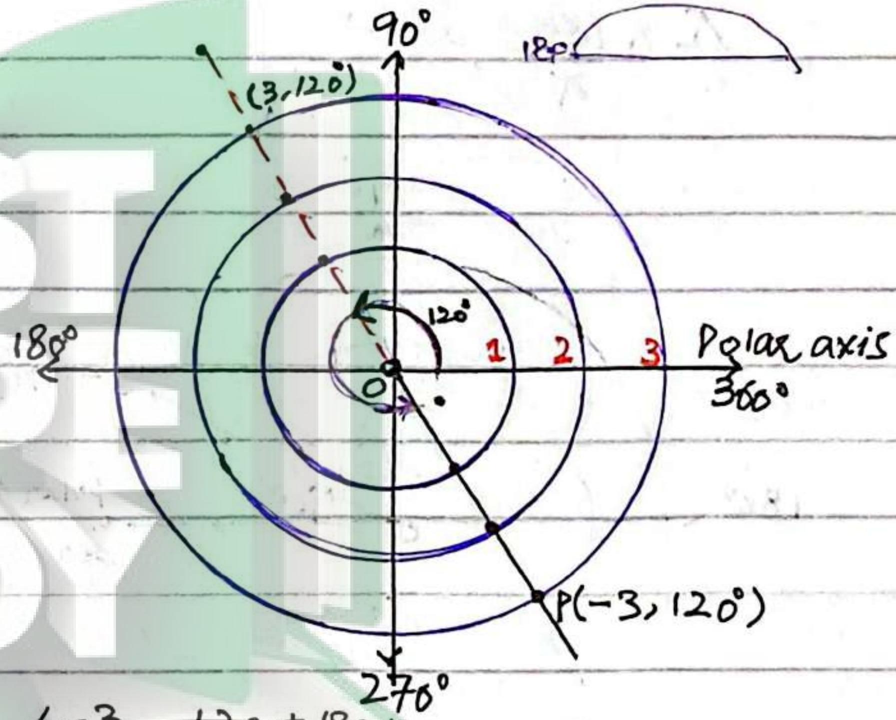
Q#1 Plot the Points

(i) $(2, 75^\circ)$



The polar Coordinates are radius is 2 units away from the pole with angle $\theta = 75^\circ$

(ii) $(-3, 120^\circ)$

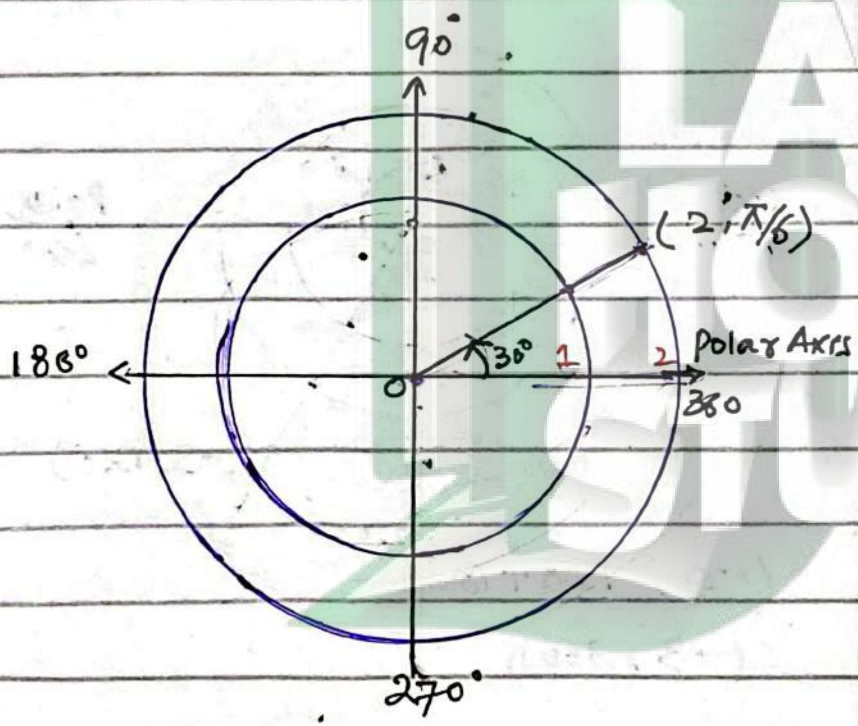


$(-3, 120 + 180)$

$(-3, 300)$

(iii) $(2, \pi/6)$ OR $(2, 30^\circ)$

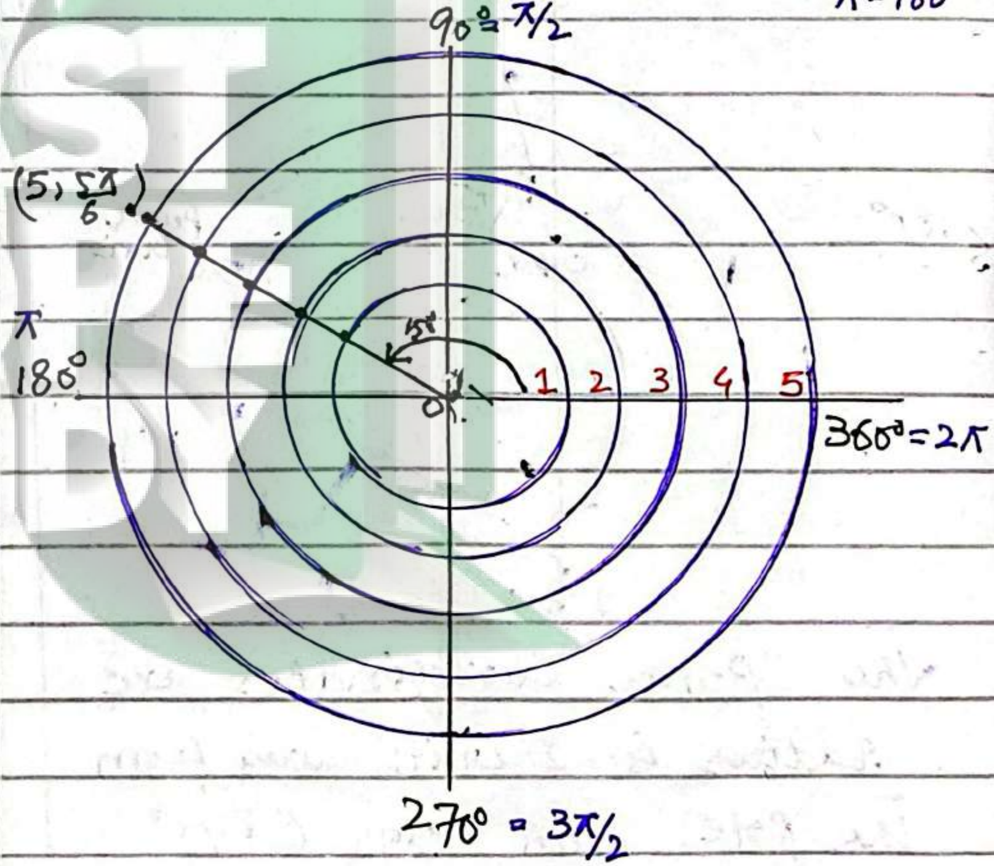
$$\frac{\pi}{6} \times \frac{180}{\pi} = 30^\circ$$



(iv) $\left(5, \frac{5\pi}{6}\right)$ OR $(5, 150^\circ)$

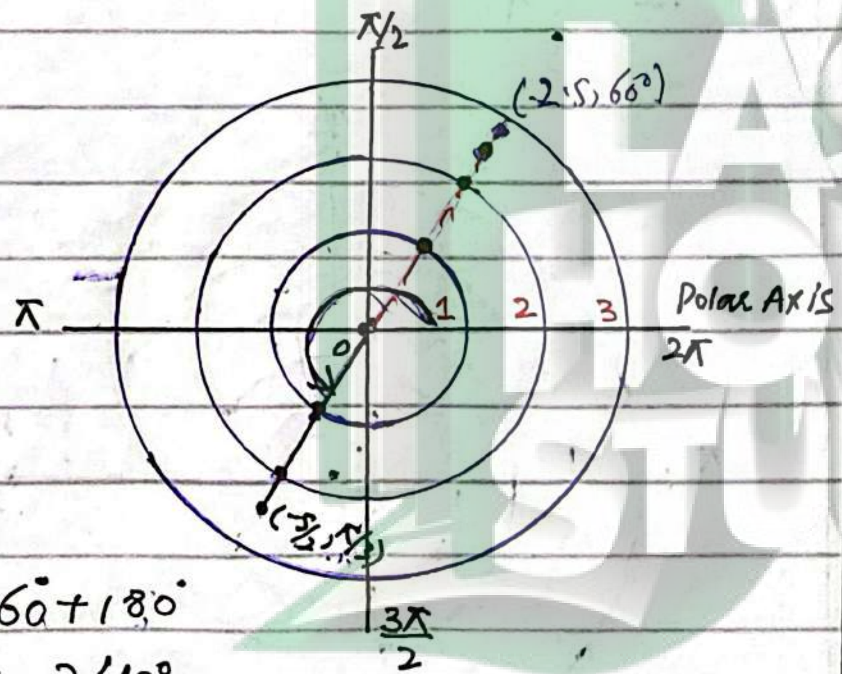
$$\frac{5\pi}{6} \times \frac{180}{\pi} = 150^\circ$$

$\pi = 180^\circ$



(V) $\left(-\frac{5}{2}, \frac{\pi}{3}\right) \Rightarrow (-2.5, 60^\circ)$

$$\frac{\pi}{3} \times \frac{180}{\pi} = 60^\circ$$



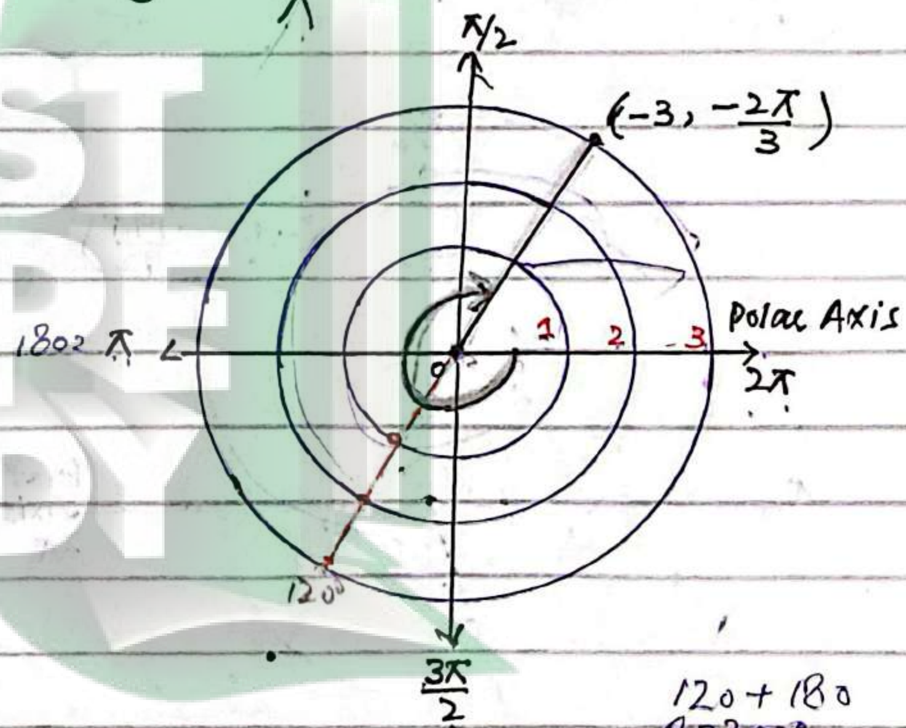
$\theta = 60^\circ + 180^\circ$

$\theta = 240^\circ$

$\left(-\frac{5}{2}, 240^\circ\right)$

(VI) $\left(-3, -\frac{2\pi}{3}\right)$

$$\frac{2\pi}{3} \times \frac{180}{\pi} = 120^\circ \qquad \pi = 180^\circ$$



$120 + 180$

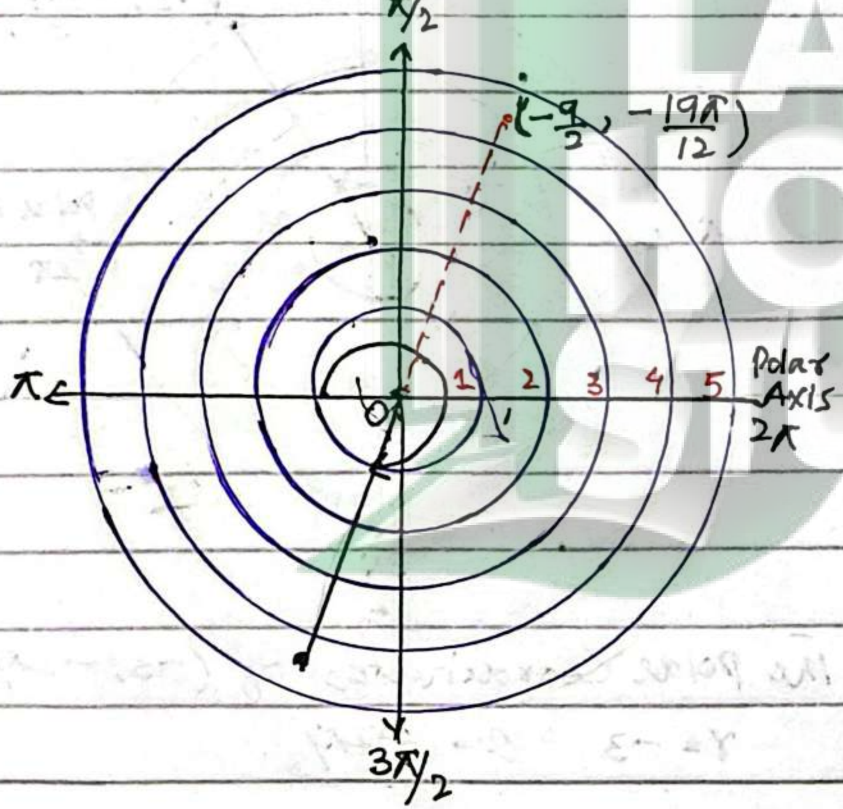
$\theta = 300^\circ$

The polar Coordinates of $(-3, -2\pi/3)$

$r = -3 \quad \theta = -2\pi/3$

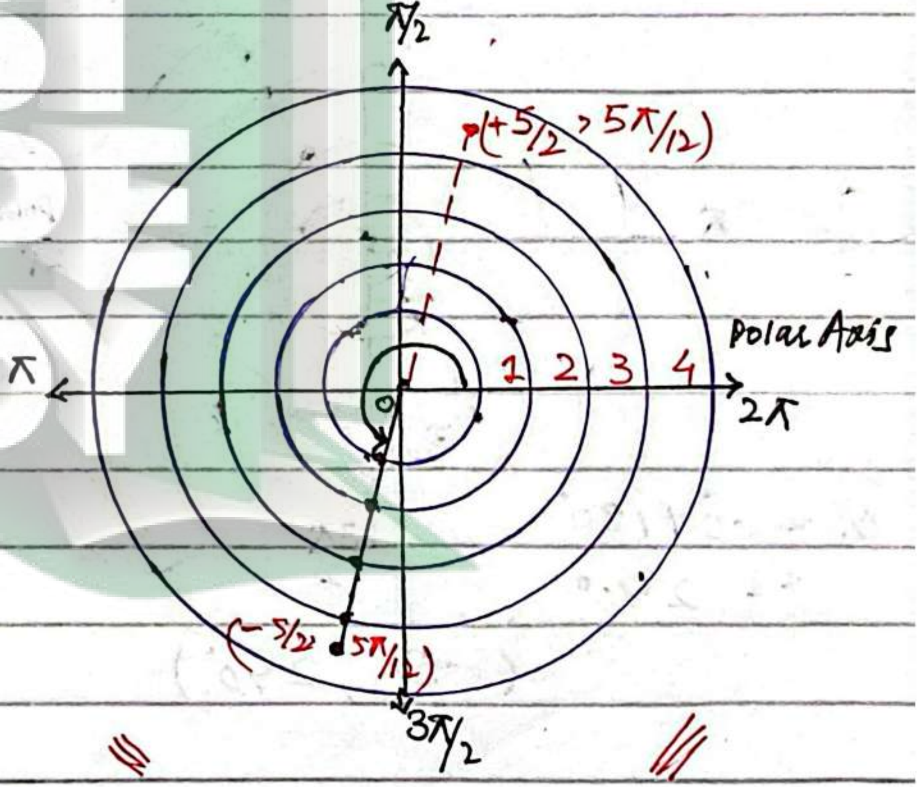
(vii) $\left(-\dfrac{9}{2}, -\dfrac{19\pi}{12}\right) \Rightarrow (-4.5, -285^\circ)$

$$-\frac{19\pi}{12} \times \frac{180}{\pi} = -285^\circ$$



(viii) $\left(-5/2, \dfrac{5\pi}{12}\right)$ OR $(-4.5, 75^\circ)$

$$\frac{5\pi}{12} \times \frac{180}{\pi} = 75^\circ$$



Q#2 Express the Complex Numbers in Polar form

(i) $4+3i$

$x = 4 \quad y = 3$

Polar form of complex number.

$r = \sqrt{x^2+y^2} \Rightarrow \sqrt{(4)^2+(3)^2}$

$r = \sqrt{16+9} \Rightarrow r = \sqrt{25} = 5$

$\theta = \tan^{-1}|y/x|$

$\theta = \tan^{-1}(3/4) \Rightarrow 36.86^\circ$

polar Form $r(\cos\theta + i\sin\theta)$

$5(\cos 36.86 + i\sin 36.86)$

(ii) $1+i$

$x = 1 \quad y = 1$

$r = \sqrt{x^2+y^2} \Rightarrow \sqrt{(1)^2+(1)^2}$

$r = \sqrt{1+1} \Rightarrow r = \sqrt{2}$

Now $\theta = \tan^{-1}(y/x)$

$\theta = \tan^{-1}(1/1)$

$\theta = \tan^{-1}(1)$

$\theta = 45^\circ$

Polar Form $r(\cos\theta + i\sin\theta)$

$\sqrt{2}(\cos 45^\circ + i\sin 45^\circ)$

(iii) $\left(\frac{1}{2} + \frac{\sqrt{3}}{2} i\right)$

$x = 1/2 \quad y = \sqrt{3}/2$

$r = \sqrt{x^2 + y^2} \Rightarrow \sqrt{\left(\frac{1}{2}\right)^2 + \left(\sqrt{3}/2\right)^2}$

$r = \sqrt{\frac{1}{4} + \frac{3}{4}} \Rightarrow r = \sqrt{\frac{1+3}{4}}$

$r = \sqrt{\frac{4}{4}} = 1$

$\theta = \tan^{-1}(y/x) \Rightarrow \tan^{-1}\left(\frac{\sqrt{3}/2}{1/2}\right)$

$\theta = \tan^{-1}\left(\frac{\sqrt{3}}{2} \times \frac{2}{1}\right) \Rightarrow \tan^{-1}(\sqrt{3})$

$\theta = \pi/3$ polar Form

$r(\cos\theta + i\sin\theta)$

$1(\cos\pi/3 + i\sin\pi/3)$

(iv) $\left(-5/2 + \frac{5\sqrt{3}}{2} i\right)$

$x = -5/2 \quad y = -\frac{5\sqrt{3}}{2}$

$r = \sqrt{x^2 + y^2} \Rightarrow \sqrt{(-5/2)^2 + \left(\frac{-5\sqrt{3}}{2}\right)^2}$

$r = \sqrt{\frac{25}{4} + \frac{75}{4}} \Rightarrow \sqrt{\frac{25+75}{4}} \Rightarrow \sqrt{\frac{100}{4}}$

$r = \frac{10}{2} = 5$

$\theta = \tan^{-1}(y/x) \Rightarrow \tan^{-1}\left(\frac{-\frac{5\sqrt{3}}{2}}{-5/2}\right)$

$\theta = \tan^{-1}\left(\frac{-5\sqrt{3}}{2} \times \frac{2}{5}\right) \Rightarrow \tan^{-1}(\sqrt{3})$

$\theta = \pi/3$

Polar form

$r(\cos\theta + i\sin\theta)$

$5(\cos \pi/3 + i \sin \pi/3)$

(V) $-\frac{5}{7} + \frac{\sqrt{8}}{7} i$

$x = -5/7 \quad y = \frac{\sqrt{8}}{7}$

$r = \sqrt{x^2 + y^2} \Rightarrow \sqrt{\left(-\frac{5}{7}\right)^2 + \left(\frac{\sqrt{8}}{7}\right)^2}$

$r = \sqrt{\frac{25}{49} + \frac{8}{49}} \Rightarrow \sqrt{\frac{25+8}{49}}$

$r = \sqrt{\frac{33}{49}} \Rightarrow r = \frac{\sqrt{33}}{7}$

$\theta = \tan^{-1}(y/x)$

$\theta = \tan^{-1}\left(\frac{\sqrt{8}}{7} \times \frac{-7}{5}\right) \Rightarrow \tan^{-1}\left(-\frac{\sqrt{8}}{5}\right)$

$\theta = \tan^{-1}(-0.5656)$

$\theta = -29.49$

$\theta = 180 - 29.49$

$\boxed{\theta = 150.5}$

Polar Form

$r(\cos\theta + i\sin\theta)$

$\frac{\sqrt{33}}{7}(\cos 150.5 + i \sin 150.5)$

(vi) $\frac{1}{2} + \frac{\sqrt{5}}{2} i$

$x = \frac{1}{2} \quad y = \frac{\sqrt{5}}{2}$

$r = \sqrt{x^2 + y^2} \Rightarrow \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{5}}{2}\right)^2}$

$r = \sqrt{\frac{1}{4} + \frac{5}{4}} \Rightarrow \sqrt{\frac{1+5}{4}}$

$r = \sqrt{\frac{6}{4}} \Rightarrow r = \frac{\sqrt{6}}{2}$

$\theta = \tan^{-1}(y/x) \Rightarrow \tan^{-1}\left(\frac{\sqrt{5}}{2} \times \frac{2}{1}\right)$

$\theta = \tan^{-1}(\sqrt{5}) \Rightarrow \tan^{-1}(2.236)$

$\boxed{\theta = 66}$

polar Form is

$r(\cos\theta + i\sin\theta)$

$\frac{\sqrt{6}}{2}[\cos 66^\circ + i\sin 66^\circ]$

(vii) $-\frac{2}{3} - \frac{\sqrt{7}}{3} i$

$x = -\frac{2}{3} \quad y = \frac{-\sqrt{7}}{3}$

$r = \sqrt{x^2 + y^2} \Rightarrow \sqrt{\left(\frac{-2}{3}\right)^2 + \left(\frac{-\sqrt{7}}{3}\right)^2}$

$r = \sqrt{\frac{4}{9} + \frac{7}{9}} \Rightarrow \sqrt{\frac{4+7}{9}}$

$r = \sqrt{\frac{11}{9}} \Rightarrow \boxed{r = \frac{\sqrt{11}}{3}}$

Polar Form

$\theta = \tan^{-1}(y/x)$

$\theta = \tan^{-1}\left(\frac{-\sqrt{7}}{3} \times -\frac{3}{2}\right)$

$\theta = \tan^{-1}(\sqrt{7}/2) \Rightarrow \tan^{-1}(1.323)$

$\boxed{\theta = 53^\circ}$

Polar Form is

$r(\cos\theta + i\sin\theta)$

$\frac{\sqrt{11}}{3}\left[\cos 53^\circ + i \sin 53^\circ\right]$

Q#3

Convert Complex numbers in the Rectangular form

$x + iy$

(i) $4\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right)$

$4\left(\frac{1}{2} + i\left(-\frac{\sqrt{3}}{2}\right)\right)$

$= \frac{4}{2} + i\left(4 \times -\frac{\sqrt{3}}{2}\right)$

$= 2 - 2\sqrt{3}i$

(ii) $\frac{3}{2}\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right)$

$= \frac{3}{2}\left[0.997 - 0.0639i\right]$

$= 1.4955 - 0.098i$

(iii) $|Z| = 7 \quad \arg(Z) = \frac{23\pi}{12}$

$|Z| = r = 7 \quad \arg = \theta = \frac{23\pi}{12}$

$r(\cos\theta + i\sin\theta)$

$= 7\left[\cos\frac{23\pi}{12} + i\sin\frac{23\pi}{12}\right]$

$= 7[0.99 + 0.105i]$

$= 6.93 + 0.734i$

(iv) $|Z| = 11 \quad \arg(Z) = \frac{-11\pi}{12}$

$r = 11 \quad \theta = \frac{-11\pi}{2}$

$\therefore r(\cos\theta + i\sin\theta)$

$= 11\left[\cos\left(\frac{-11\pi}{2}\right) + i\sin\left(\frac{-11\pi}{2}\right)\right]$

$= 11(0.955 - 0.297i)$

$= 10.505 - 3.267i$

(v)

$|Z| = \frac{10}{3} \quad \arg(Z) = \frac{-17\pi}{12}$

$r = \frac{10}{3} \quad \theta = \frac{-17\pi}{2}$

$r(\cos\theta + i\sin\theta)$

$= \frac{10}{3}\left[\cos\left(\frac{-17\pi}{2}\right) + i\sin\left(\frac{-17\pi}{2}\right)\right]$

$= \frac{10}{3}(0.893 - 0.449i)$

$= 2.97 - 1.497i$ Ans

(vi) $2\cos(-33) + i\,2\sin(-33)$

$= 2[\cos(-33) + i\sin(-33)]$

$= 2[0.8386 - 0.5446i]$

$= 1.677 - 1.0893i$

(vii) $|Z| = 12 \quad \arg(Z) = \pi$

$r = 12 \quad \theta = \pi \Rightarrow \theta = 180^\circ$

$r(\cos\theta + i\sin\theta)$

$12[\cos\pi + i\sin\pi]$

$= 12[\,\cancel{\ldots}\ -1 + i(0)]$

$-12 + 0i$ Ans

Q#4 if $Z_1 = 9\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)$

$Z_2 = 5(\cos\pi/3 + i\sin\pi/3)$

Then find

(i) $Z_1 + Z_2$

$9\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right) + 5\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

$= 9(\cos 225^\circ + i\sin 225^\circ) + 5(\cos 60^\circ + i\sin 60^\circ)$

$= 9(-0.707 - 0.707i) + 5(0.5 + 0.866i)$

$= -6.36 - 6.36i + 2.5 + 4.33i$

$= -3.86 - 2.03i$ Ans

(ii) $Z_1 - Z_2$

$= 9\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right) - 5\left(\cos\pi/3 + i\sin\pi/3\right)$

$= 9\left(\cos 225^\circ + i\sin 225^\circ\right) - 5\left(\cos 60^\circ + i\sin 60^\circ\right)$

$= 9(-0.707 - 0.707i) - 5(0.5 + 0.866i)$

$= \cancel{9}\,(-6.3639 - 6.3639i) - (2.5 + 4.33i)$

$-6.3639 - 6.3639i - 2.5 - 4.33i$

$= -8.86 - 10.69i$ Ans.

Q#4(iii) $Z_1 \cdot Z_2$

Formula $r_1 r_2 [\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]$

$r_1 = 9$, $r_2 = 5$, $\theta_1 = 5\pi/4$, $\theta_2 = \pi/3$

$$= 9 \times 5 \left[\cos\left(\frac{5\pi}{4}+\frac{\pi}{3}\right) + i\sin\left(\frac{5\pi}{4}+\frac{\pi}{3}\right)\right]$$

$$= 45 \left[\cos\left(\frac{15\pi+4\pi}{12}\right) + i\sin\left(\frac{15\pi+4\pi}{12}\right)\right]$$

$$= 45 \left[\cos\frac{19\pi}{12} + i\sin\frac{19\pi}{12}\right] \text{ Ans}$$

(iv) Z_1/Z_2

Formula: $\frac{r_1}{r_2}[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)]$

$$= \frac{9}{5}\left[\cos\left(\frac{5\pi}{4}-\frac{\pi}{3}\right) + i\sin\left(\frac{5\pi}{4}-\frac{\pi}{3}\right)\right]$$

$$= 9/5\left[\cos\frac{11\pi}{12} + i\sin\frac{11\pi}{12}\right] \text{ Ans}$$

Q#5 if $Z_1 = 7\left(\cos\frac{23\pi}{12} + i\sin\frac{23\pi}{12}\right)$ $Z_2 = 11\left(\cos\frac{11\pi}{12} + i\sin\frac{11\pi}{12}\right)$

Then find the following and express into $x+iy$ form

(i) $Z_1 + Z_2$

$$= 7\left(\cos\frac{23\pi}{12} + i\sin\frac{23\pi}{12}\right) + 11\left(\cos\frac{11\pi}{12} + i\sin\frac{11\pi}{12}\right)$$

$$= 7(0.9659 + i\,0.259) + 11(-0.965 + 0.259i)$$

$$= 6.76 - 1.813i - 10.615 + 2.849i$$

$$(-3.86 + 1.03i) \text{ Ans}$$

(ii) $Z_1 - Z_2$

$$= 7\left(\cos\frac{23\pi}{12} + i\sin\frac{23\pi}{12}\right) - 11\left(\cos\frac{11\pi}{12} + i\sin\frac{11\pi}{12}\right)$$

$$= (6.76 - 1.813i) - (-10.615 + 2.8491i)$$

$$= 6.76 - 1.813i + 10.615 - 2.849i$$

$$= (17.38 - 4.651i) \text{ Ans}$$

(iii) $Z_1 \cdot Z_2$ $\quad r_1 = 7 \quad r_2 = 11 \quad \theta_1 = \frac{23\pi}{12} \quad \theta_2 = \frac{11\pi}{12}$

$$= 7 \times 11 \left[\cos\left(\frac{23\pi}{12} + \frac{11\pi}{12}\right) + i \sin\left(\frac{23\pi}{12} + \frac{11\pi}{12}\right)\right]$$

$$= 77 \left[\cos\frac{34\pi}{12} + i \sin\frac{34\pi}{12}\right]$$

$$= 77 \left[\cos\frac{17\pi}{6} + i \sin\frac{17\pi}{6}\right]$$

$$= 77(-0.866 + 0.5i)$$

$$= (-66.68 + 38.5i) \text{ Ans}$$

(iv) Z_1 / Z_2

$$= \frac{7}{11}\left[\cos\left(\frac{23\pi}{12} - \frac{11\pi}{12}\right) + i \sin\left(\frac{23\pi}{12} - \frac{11\pi}{12}\right)\right.$$

$$\frac{7}{11}\left[\cos\left(\frac{12}{12}\right) + i\sin\left(\frac{12}{12}\right)\right] \Rightarrow \frac{7}{11}\left[0.998 + i(0.02)\right]$$

$$= \frac{7}{11}\{1 + i(0)\} \Rightarrow \frac{7}{11} + 0i \text{ Ans}$$

Q#6 if Z_1 and Z_2 are two complex numbers show that

(i) $Arg(Z_1Z_2) = Arg Z_1 + Arg Z_2$

Let $Z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $Z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$Arg Z_1 = \theta_1$ and $Arg Z_2 = \theta_2$

L.H.S $Arg(Z_1Z_2)$

$(Z_1Z_2) = r_1(\cos\theta_1 + i\sin\theta_1) \cdot r_2(\cos\theta_2 + i\sin\theta_2)$

$(Z_1Z_2) = r_1 \cdot r_2[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]$

$Arg(Z_1Z_2) = \theta_1 + \theta_2$

R.H.S $Arg Z_1 + Arg Z_2$

$\theta_1 + \theta_2$ Proved

Q#6(ii) $Arg(Z_1/Z_2) = Arg\,Z_1 - Arg\,Z_2$

Let $Z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $Z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$Arg\,Z_1 = \theta_1$ and $Arg\,Z_2 = \theta_2$

L.H.S $Arg(Z_1/Z_2)$

$$\frac{Z_1}{Z_2} = \frac{r_1}{r_2}\left[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right]$$

$$Arg\left(\frac{Z_1}{Z_2}\right) = \theta_1 - \theta_2$$

R.H.S $Arg\,Z_1 - Arg\,Z_2$

$\theta_1 - \theta_2$

L.H.S = R.H.S

Q#7 Divide $Z_1 = 6(\cos 150^\circ + i\sin 150^\circ)$ by $Z_2 = 3(\cos 30^\circ + i\sin 30^\circ)$ and express in $x+iy$ form.

Formula : $\frac{Z_1}{Z_2} = \frac{r_1}{r_2}\left[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)\right]$

$\Rightarrow$ Here $r_1 = 6$ $r_2 = 3$ $\theta_1 = 150^\circ$ $\theta_2 = 30^\circ$

$$\frac{Z_1}{Z_2} = \frac{6}{3}\left[\cos(150^\circ-30^\circ) + i\sin(150^\circ-30^\circ)\right]$$

$$= 2\left[\cos 120^\circ + i\sin 120^\circ\right] \text{ OR } (-1+\sqrt{3}i) \text{ Ans}$$

Q#8 Multiplying $Z_1 = 2(\cos 60^\circ + i\sin 60^\circ)$ and $Z_2 = 5(\cos 90^\circ + i\sin 90^\circ)$ and express in $x+iy$ form :-

Formula

$$Z_1 \cdot Z_2 = r_1 \cdot r_2\left[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\right]$$

$$= 2\times 5\left[\cos(60+90) + i\sin(60+90)\right] \Rightarrow 10\left[\cos 150^\circ + i\sin 150^\circ\right] \text{ OR}$$

$$= -5\sqrt{3} + 5i$$

Q#9 Find the modulus and argument of $Z = -2-2i$

$Z = -2-2i$

Modulus $|Z| = \sqrt{(-2)^2+(-2)^2}$

$|Z| = \sqrt{4+4} \Rightarrow \sqrt{8} \Rightarrow 2\sqrt{2}$

Now argument

$Arg(Z) = \tan^{-1}\left(\frac{-2}{-2}\right)$

$= \tan^{-1}(1) \Rightarrow \pi/4$

$\text{Argument}(Z) = \frac{\pi}{4} + \pi$

$= \frac{\pi + 4\pi}{4}$

$= \frac{5\pi}{4} + 2n\pi$ Ans

Q#10 Write the Equation $\arg(\bar{Z} - 2 + i) = \frac{2\pi}{3}$ to Cartiesian form if $Z = x+iy$

Let $Z = x+iy$ $\bar{Z} = x-iy$

$Arg(x - iy - 2 + i) = 2\pi/3$

$Arg(x - 2 + i - iy) = 2\pi/3$

$Arg[(x-2) + (1-y)i] = 2\pi/3$

$Arg(Z) \Rightarrow \tan^{-1}\left(\frac{1-y}{x-2}\right) = \frac{2\pi}{3}$

$\frac{1-y}{x-2} = \tan\left(\frac{2\pi}{3}\right) \Rightarrow \frac{1-y}{x-2} = -\sqrt{3}$

$1-y = -\sqrt{3}(x-2) \Rightarrow 1-y = -\sqrt{3}x + 2\sqrt{3}$

$y = \sqrt{3}x - 2\sqrt{3} + 1$ Ans

Q#11 if $Z = x + iy$ and $Arg\left(\frac{\bar{Z}-1+2i}{\bar{Z}+1-2i}\right) = \frac{9\pi}{4}$

Show that $x^2 + y^2 - 4x + 2y - 5 = 0$

We know that $Arg\left(\frac{Z_1}{Z_2}\right) = Arg(Z_1) - Arg(Z_2)$

$$Arg(\bar{Z}-1+2i) - Arg(\bar{Z}+1-2i) = \frac{9\pi}{4}$$

given that $Z = x+iy$ and $\bar{Z} = x - iy$

$$Arg(x-iy-1+2i) - Arg(x-iy+1-2i) = \frac{9\pi}{4}$$

$$Arg[(x-1)-(y+2)i] - Arg[(x+1)-(y+2)i] = 9\pi/4$$

$$\tan^{-1}\left(\frac{-(y-2)}{(x-1)}\right) - \tan^{-1}\left(\frac{-(y+2)}{x+1}\right) = \frac{9\pi}{4}$$

$$\tan^{-1}A - \tan^{-1}B = \tan^{-1}\left[\frac{A-B}{1+AB}\right]$$

$$\tan^{-1}\left(\frac{\frac{2-y}{x-1} - \frac{(-2-y)}{x+1}}{1+\left(\frac{2-y}{x-1}\right)\left(\frac{-2-y}{x+1}\right)}\right) = \frac{9\pi}{4}$$

$$\left[\frac{\dfrac{(2-y)(x+1)-(-2-y)(x-1)}{(x+1)(x-1)}}{\dfrac{(x+1)(x-1)+(2-y)(-2-y)}{(x+1)(x-1)}}\right] = \tan\left(\frac{9\pi}{4}\right)$$

$$\frac{2x+\cancel{2}-\cancel{yx}-y+2x-\cancel{2}+\cancel{yx}-y}{x^2-1+(-4)-\cancel{2y}+\cancel{2y}+y^2} = 1$$

$$\frac{4x-2y}{x^2+y^2-5} = \frac{1}{1}$$

$$x^2+y^2-5 = 4x-2y$$

$$x^2+y^2-4x+2y-5=0 \text{ Ans.}$$

Q#12 if $Z = x+iy$ and $\arg(Z-2-3i) - \arg(Z+2+3i) = 2\pi$

Show that $2y = 3x$

$$\operatorname{Arg}(x+iy-2-3i) - \arg(x+iy+2+3i) = 2\pi$$

$$\arg[(x-2)+(y-3)i] - \arg[(x+2)+(y+3)i] = 2\pi$$

$$\tan^{-1}\left[\frac{y-3}{x-2}\right] - \tan^{-1}\left[\frac{y+3}{x+2}\right] = 2\pi$$

Formula: $\tan^{-1}\left[\frac{A-B}{1+AB}\right] \Rightarrow \tan^{-1}\left[\frac{\frac{y-3}{x-2} - \frac{y+3}{x+2}}{1+\frac{y-3}{x-2}\times\frac{y+3}{x+2}}\right] = 2\pi$

$$\tan^{-1}\left[\frac{\frac{(y-3)(x+2)-(y+3)(x-2)}{(x-2)(x+2)}}{\frac{(x-2)(x+2)+(y-3)(y+3)}{(x-2)(x+2)}}\right] = 2\pi$$

$$\tan^{-1}\left[\frac{\dfrac{(xy+2y-3x-6)-(xy-2y+3x-6)}{(x-2)(x+2)}}{\dfrac{(x^2-4)+y^2-9}{(x-2)(x+2)}}\right]=2\pi$$

$$\left[\frac{\cancel{xy}+2y-3x-6-\cancel{xy}+2y-3x+6}{x^2-4+y^2-9}\right]=\tan(2\pi)$$

$$\frac{4y-6x}{x^2+y^2-13}=0$$

$$4y-6x=0$$

$$2(2y-3x)=0$$

$$2y-3x=0$$

$$2y=3x \quad \underline{\text{proved}}$$

(13) Solve the equation $|Z-2i| = |\bar{Z}+2|$ for

$Z = x+iy \qquad \bar{Z} = x-iy$

$$|x+iy-2i| = |x-iy+2|$$

$$|x+(y-2)i| = |(x+2)+(-y)i|$$

$$\sqrt{x^2+(y-2)^2} = \sqrt{(x+2)^2+(-y)^2}$$

$$\sqrt{x^2+y^2+4-4y} = \sqrt{x^2+4+4x+y^2}$$

$$\cancel{x^2}+\cancel{y^2}+\cancel{4}-4y = \cancel{x^2}+\cancel{4}+4x+\cancel{y^2}$$

$$-4y = 4x$$

$$\boxed{-y = x} \text{ Ans}$$

Q#14 For $z = x+iy$, Solve the equation $|5z+4+i|$

$= |5\bar{z} - 3 + 2i| \qquad \bar{z} = x - iy$

$|5(x+iy)+4+i| = |5(x-iy)-3+2i|$

$|5x+5yi+4+i| = |5x-5yi-3+2i|$

$|(5x+4)+(5y+1)i| = |(5x-3)+(2-5y)i|$

$\sqrt{(5x+4)^2+(5y+1)^2} = \sqrt{(5x-3)^2+(2-5y)^2}$

$\cancel{25x^2}+16+40x+\cancel{25y^2}+1+10y = \cancel{25x^2}+9-30x+4+\cancel{25y^2}-20y$

$40x+30x+10y+20y+17-13=0$

$70x+30y+4=0 \Rightarrow 35x+15y+2=0 \Rightarrow 15y=-35x-2$

$y = -35/15\,x - \frac{2}{15} \Rightarrow y = -7/3\,x - 2/15$ Ans.

Q#15 Determine the set of points $z = x+iy$ that satisfy the equation $|3\bar{z} - 2 + i| = |3z + i|$.

$z = x+iy$ and $\bar{z} = x - iy$

$$|3(x-iy) - 2 + i| = |3(x+iy) + i|$$

$$|3x - 3yi - 2 + i| = |3x + 3yi + i|$$

$$|(3x-2) + (1-3y)i| = |3x + (3y+1)i|$$

$$\sqrt{(3x-2)^2 + (1-3y)^2} = \sqrt{(3x)^2 + (3y+1)^2}$$

$$\cancel{9x^2} + 4 - 12x + \cancel{1} + \cancel{9y^2} - 6y = \cancel{9x^2} + \cancel{9y^2} + \cancel{1} + 6y$$

$$-12x - 6y - 6y + 4 = 0 \Rightarrow -12x - 12y + 4 = 0$$

$$-3x - 3y + 1 = 0 \Rightarrow 3y = -3x + 1$$

$$y = -\tfrac{3}{3}x + \tfrac{1}{3} \Rightarrow \boxed{y = -x + \tfrac{1}{3}} \text{ Ans}$$

(16) if $Z = x + iy$ and $W = \dfrac{1 - iZ}{Z - i}$ show that $|W| = 1$ Z is real.

$$W = \frac{1 - i(x)}{x - i}$$

$$W = \frac{1 - xi}{x - i} \times \frac{x + i}{x + i}$$

$$W = \frac{x + i - x^2 i - x i^2}{x^2 - i^2}$$

$$W = \frac{x + i - x^2 i + x}{x^2 - (-1)}$$

$$W = \frac{2x + (1 - x^2)i}{x^2 + 1}$$

$$W = \frac{2x}{x^2+1} + \frac{(1 - x^2)i}{x^2+1}$$

$|W| = 1$ (Given)

$$|W| = \left| \frac{2x}{x^2+1} + \frac{(1 - x^2)i}{x^2+1} \right|$$

$$|W| = \sqrt{\left(\frac{2x}{x^2+1}\right)^2 + \left(\frac{1 - x^2}{x^2+1}\right)^2}$$

$$= \sqrt{\frac{4x^2}{(x^2+1)^2} + \frac{(1 - x^2)^2}{(x^2+1)^2}}$$

$$= \sqrt{\frac{4x^2 + (1 - x^2)^2}{(x^2+1)^2}}$$

$$= \sqrt{\frac{4x^2+1+x^4-2x^2}{(x^2+1)^2}}$$

$$= \sqrt{\frac{(x^4+2x^2+1)}{(x^4+2x^2+1)}}$$

$$|W| = \sqrt{1}$$

$$|W| = 1 \quad \underline{\underline{\text{proved}}}$$

Q#17 if Z_1 and Z_2 are different complex numbers with $|Z_2| = 1$, find

$$\left|\frac{Z_2 - Z_1}{1 - \bar{Z}_1 Z_2}\right|$$

Q #18 An AC source supplies a voltage of $V = 120\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$ volts to a circuit with impedance $Z = \frac{1+\sqrt{3}i}{2}$ ohms. Calculate the current in polar form.

$V = 120\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

$V \propto I$

Ohms Law $V = ZI$ $V = RI$

$V = 120\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right), Z = \left(\frac{1+\sqrt{3}i}{2}\right) I = ?$

$r = \sqrt{(1/2)^2 + (\sqrt{3}/2)^2} = \sqrt{\frac{1}{4} + \frac{3}{4}} = \sqrt{\frac{4}{4}}$

$\boxed{r = 1}$ $\theta = \tan^{-1}\left(\frac{\sqrt{3}/2}{1/2}\right)$

$\theta = \tan^{-1}(\sqrt{3}) \Rightarrow \theta = \pi/3$

$Z = r(\cos\theta + i\sin\theta)$

$Z = 1(\cos\pi/3 + i\sin\pi/3)$

$I = \frac{V}{Z}$

$I = \frac{r_1}{r_2}\left[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right]$

$r_1 = 120$ $r_2 = 1$ $\theta_1 = \pi/4$ $\theta_2 = \frac{\pi}{3}$

$I = \frac{120}{1}\left[\cos\left(\frac{\pi}{4} - \frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3} - \frac{\pi}{3}\right)\right]$

$I = 120\left[\cos\left(-\frac{\pi}{12}\right) + i\sin\left(-\frac{\pi}{12}\right)\right]$

$I = 120\left[\cos\frac{\pi}{12} - i\sin\frac{\pi}{12}\right]$

$D = \frac{16+4}{5} \Rightarrow D = \frac{20}{5} = 4$

$E = \frac{Z_4}{K} = (10-5i)\left(\frac{2+i}{5}\right)$

$E = \frac{20+10i-10i-5i^2}{5} \Rightarrow \frac{20+5}{5}$

$E = \frac{25}{5} \Rightarrow E = 5$

Q#21 Consider the complex ~~num~~ encryption key $K = 3-3i$. Encrypt the word "QUIZ" and then decrypt it back to the original word using the inverse of the key.

Sol.- $K = 3-3i$

$\frac{1}{K} = \frac{1}{3-3i} \times \frac{3+3i}{3+3i}$

$= \frac{3+3i}{(3)^2-(3i)^2} \Rightarrow \frac{3+3i}{9-9i^2}$

$\frac{1}{K} = \frac{3+3i}{9+9} = \frac{3+3i}{18}$

$= \frac{3(1+i)}{18} \Rightarrow \frac{1+i}{6}$

$Q = 17+0i$ $U = 21+0i$ $I = 9+0i$

$Z = 26+0i$

Encryption of Q, U, I, Z

$Z_1 = K \times Q \Rightarrow 17(3+3i) = 51+51i$

$Z_2 = K \times U \Rightarrow 21(3+3i) = 63+63i$

$Z_3 = K \times I = 9(3+3i) = 27+27i$

$Z_4 = K \times Z = \cancel{26}(78+78i) = 78+78i$

Decryption

$Q = \frac{Z_1}{K} = \frac{(51-51i)(1+i)}{6}$

A=1	E=5	I=9	M=13	Q=17	U=21	Y=25
B=2	F=6	J=10	N=14	R=18	V=22	Z=26
C=3	G=7	K=11	O=15	S=19	W=23	
D=4	H=8	L=12	P=16	T=20	X=24	

Q #20 Encrypt the word "CODE" by multiplying the complex encryption key $K = 2-i$ then decrypt it back to the original word.

Sol:- $K = 2-i$ $C=3$ $O=15$ $D=4$ $E=5$.

$C = 3+0i$, $O = 15+0i$ $D = 4+0i$ $E = 5+0i$.

Encryption $Z_1 = K\times C$

$Z_1 = (2-i)(3+0i) \Rightarrow 6-3i$

$Z_2 = K\times O \Rightarrow (2-i)(15+0i) = 30-15i$

$Z_3 = K\times D \Rightarrow (2-i)(4+0i) = 8-4i$

$Z_4 = K\times E \Rightarrow (2-i)(5+0i) = 10-5i$

Back to the original word.

$$\frac{1}{K} = \frac{1}{2-i}\times\frac{2+i}{2+i} \Rightarrow \frac{2+i}{(2)^2-(i)^2}$$

$$\frac{1}{K} = \frac{2+i}{4-i^2} \Rightarrow \frac{2+i}{4+1} \Rightarrow \frac{2+i}{5}$$

$$C = \frac{Z_1}{K} \Rightarrow (6-3i)\left(\frac{2+i}{5}\right) \Rightarrow \frac{12+6i-6i-3i^2}{5}$$

$$C = \frac{12+3}{5} \Rightarrow \frac{15}{5} = 3$$

$$O = \frac{Z_2}{K} = (30-15i)\left(\frac{2+i}{5}\right).$$

$$O = \frac{60+30i-30i-15i^2}{5} = \frac{60+15}{5}$$

$$O = \frac{75}{5} = 15$$

$$D = \frac{Z_3}{K} = (8-4i)\left(\frac{2+i}{5}\right)$$

$$D = \frac{16+8i-8i-4i^2}{5}$$

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An AC circuit has an impedance of $Z = 3-6i$ ohms is connected to a voltage source of $V = 90+30i$ volts: Find the current in both rectangular and polar form.

Sol:- $Z = 3-6i$ $V = 90+30i$

Formula $V = ZI \Rightarrow I = \frac{V}{Z}$

$I = \frac{90+30i}{3-6i} \Rightarrow \frac{3(30+10i)}{3(1-2i)}$

$I = \frac{30+10i}{1-2i} \times \frac{1+2i}{1+2i}$

$= \frac{(30+10i)(1+2i)}{(1-2i)(1+2i)}$

$= \frac{30+60i+10i+20i^2}{(1)^2-(2i)^2}$

$I = \frac{30+70i-20}{1-4i^2}$

$I = \frac{10+70i}{1+4} \Rightarrow \frac{10+70i}{5}$

$I = \frac{10}{5} + \frac{70}{5}i \Rightarrow 2+14i$

$r = \sqrt{(2)^2+(14)^2} \Rightarrow \sqrt{4+196}$

$r = \sqrt{200} \Rightarrow r = 10\sqrt{2}$

$\theta = \tan^{-1}\left(\frac{14}{2}\right) \Rightarrow \tan^{-1}(7)$

$\boxed{\theta = 81.86^\circ}$ $r(\cos\theta + i\sin\theta)$

$I = 10\sqrt{2}\left[\cos 81.86 + i\sin 81.86\right]$

$$Q = \frac{51+51i-51i-51i^2}{6}$$

$$Q = \frac{51+51}{6} = \frac{102}{6} = 17$$

$$U = \frac{Z_2}{K} \Rightarrow \frac{(63-63i)(1+i)}{6}$$

$$U = \frac{63+63i-63i-63i^2}{6}$$

$$U = \frac{63+63}{6} \Rightarrow \frac{126}{6} = 21$$

$$I = \frac{Z_3}{K} = \frac{(27-27i)(1+i)}{6}$$

$$I = \frac{27+27i-27i-27i^2}{6}$$

$$I = \frac{27+27}{6} \Rightarrow I = \frac{54}{6} = 9$$

$$Z = \frac{Z_4}{K} = \frac{(78-78i)(1+i)}{6}$$

$$Z = \frac{78+78i-78i-78i^2}{6}$$

$$Z = \frac{78+78}{6} \Rightarrow \frac{156}{6} \Rightarrow 26$$

Q#22 Encrypt the word "CLASS" by adding the complex encryption key $K = -3+4i$ then decrypt it back to the original word.

Sol:- $K = -3+4i$ $\quad C = 3 \quad L = 12 \quad A = 1 \quad S = 19 \quad S = 19$

Encryption $\quad C = 3+0i \quad L = 12+0i \quad A = 1+0i \quad S = 19+0i$

$Z_1 = C+K = 3+0i+4i-3 \Rightarrow 4i$

$Z_2 = L+K = 12+0i+4i-3 = 9+4i$

$Z_3 = A+K = 1+0i+4i-3 = -2+4i$

$Z_4 = S+K = 19+0i+4i-3 = 16+4i$

Decryption

$C = Z_1 - K = 4i-(-3+4i) = 3$

$L = Z_2 - K = 4i+9-(-3+4i)$

$= 4i+9+3-4i$

$= 12$

$A = Z_3 - K = -2+4i-(-3+4i)$

$A = -2+4i+3-4i = 1$

$S = Z_4 - K = 16+4i-(-3+4i)$

$S = 16+4i+3-4i$

$S = 19$