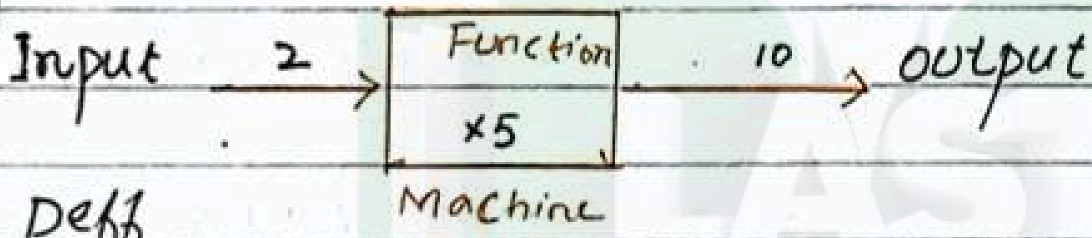
Chapter 2 Function and Graph

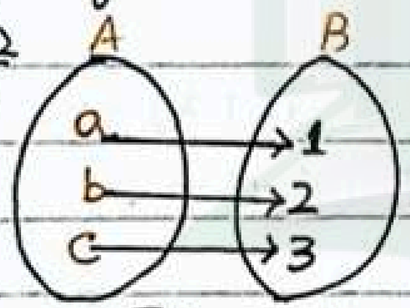
Function:-



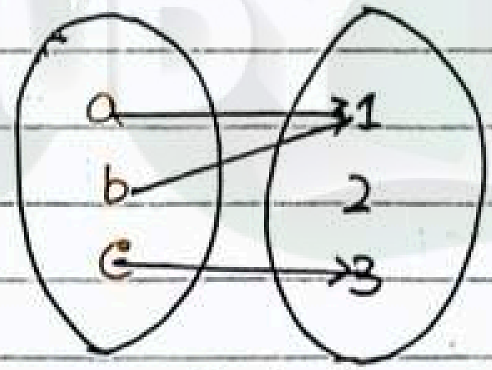
Input $\rightarrow$ Domain
output $\rightarrow$ Range
$y = f(x)$
Domain is x and Range is y.

Deff
Let A and B are two non empty sets. The relation which assign the element of A to unique element of B is called a function.

Exp



Function
"unique element"



Not a function

$f(x) = 2x$
put $x = 1, 2, 3$
$f(1) = 2(1) = \underline{\underline{2}}$
$f(2) = 2(2) = \underline{\underline{4}}$
$f(3) = 2(3) = \underline{\underline{6}}$
'unique' Solution

Exercise 2.1

Q#1:- Given that (a) $f(x) = x^2 - 1$ (b) $f(x) = \sqrt{2x+3}$

Find (i) $f(-3)$ (ii) $f(0)$ (iii) $f(x-2)$ (iv) $f(x^2+3)$

Sol:- (a) $f(x) = x^2 - 1$ —(i)

(i) $f(-3)$

put $x = -3$ in (i)

$f(-3) = (-3)^2 - 1 \Rightarrow 9 - 1 = 8$

(ii) $f(0)$

put $x = 0$ in (i)

$f(0) = (0)^2 - 1 = -1$

(iii) $f(x-2)$

put $x = x-2$ in (i)

$f(x-2) = (x-2)^2 - 1$

$f(x-2) = x^2 + 4 - 4x - 1$

$f(x-2) = x^2 - 4x + 3$

(iv) $f(x^2+3)$

put $x = x^2 + 3$ in (i)

$f(x^2+3) = (x^2+3)^2 - 1$

$= x^4 + 9 + 6x^2 - 1$

$= x^4 + 6x^2 + 8$

(b) $f(x) = \sqrt{2x+3}$ —(i)

(i) $f(-3)$

put $x=-3$ in (i)

$f(-3) = \sqrt{2(-3)+3}$

$f(-3) = \sqrt{-6+3} \Rightarrow \sqrt{-3}$

(ii) $f(0)$ put $x=0$ in (i)

$f(0) = \sqrt{2(0)+3} \Rightarrow \sqrt{3}$

(iii) $f(x-2)$

put $x = x-2$ in (i)

$f(x-2) = \sqrt{2(x-2)+3}$

$\sqrt{2x-4+3} \Rightarrow \sqrt{2x-1}$

(iv) $f(x^2+3)$

put $x = x^2+3$ in (i)

$f(x^2+3) = \sqrt{2(x^2+3)+3}$

$f(x^2+3) = \sqrt{2x^2+6+3}$

$= \sqrt{2x^2+9}$

Q#2 Find $\dfrac{f(a+h)-f(a)}{h} \rightarrow A$ and Simplify where

(i) $f(x) = 4x+7$ —— (i)

put $x = a+h$ in (i)

$f(a+h) = 4(a+h)+7$

Now put $x = a$ in (i)

$f(a) = 4a+7$

putting in Eq (A)

$$= \frac{[4(a+h)+7]-(4a+7)}{h}$$

$$= \frac{4a+4h+7-4a-7}{h}$$

$$= \frac{4h}{h} = 4 \text{ Ans}$$

(ii) $f(x) = \sin x$ —— (i)

put $x = a+h$ in (i)

$f(a+h) = \sin(a+h)$

Now put $x = a$ in (i)

$f(a) = \sin a$

putting in Eq (A)

$$= \frac{\sin(a+h) - \sin a}{h}$$

Formula:-

$$\sin\alpha - \sin\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$$

$$= 2\cos\left[\frac{a+h+a}{2}\right]\sin\left[\frac{a+h-a}{2}\right]/h$$

$$= 2\cos(a+h/2)\sin(h/2)/h$$

Q#2 (iii) $f(x) = x^3 + x^2 - 1$ —(i)

$$\frac{f(a+h) - f(a)}{h} \quad \text{—(A)}$$

put $x = a+h$ in (i)

$f(a+h) = (a+h)^3 + (a+h)^2 + 1$

Now put $x = a$ in (i)

$f(a) = a^3 + a^2 - 1$

putting in Eq (A)

$$= \frac{[(a+h)^3 + (a+h)^2 + 1] - [a^3 + a^2 - 1]}{h}$$

$\Rightarrow a^3 + 3a^2h + 3ah^2 + h^3 + a^2 + h^2 + 2ah$

$\Rightarrow +1 - a^3 - a^2 + 1/h$

$$\Rightarrow \frac{3a^2h + 3ah^2 + h^3 + h^2 + 2ah}{h}$$

$$= \frac{h[3a^2 + 3ah + h^2 + h + 2a]}{h}$$

$$= 3a^2 + 3ah + h^2 + h + 2a$$

(iv) $f(x) = \tan x$ —(i)

put $x = a+h$ in (i)

$f(a+h) = \tan(a+h)$

Now put $x=a$ in (i)

$f(a) = \tan a$

putting in (A)

$$= \frac{\tan(a+h) - \tan a}{h}$$

$$= \frac{\frac{\sin(a+h)}{\cos(a+h)} - \frac{\sin a}{\cos a}}{h}$$

$$= \frac{1}{h}\left[\frac{\sin(a+h)\cos a - \cos(a+h)\sin a}{\cos(a+h)\cos a}\right]$$

Formulae:-

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

$$= \frac{\sin(a+h-a)}{h\cos a\cos(a+h)}$$ Ans.

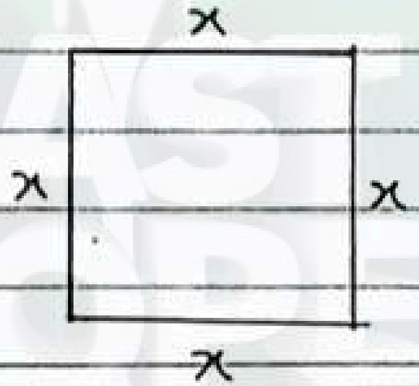
$$= \frac{\sin h}{h\cos a\cos(a+h)}$$

Q#3 Express the following.

(a) The area of a square as function of its perimeter P.

Sol:- $A = x^2$ —(i); $P = 4x$

$\frac{P}{4} = x$ putting in (i)



Area $= L \times B$
$= x \times x = x^2$
$P = x + x + x + x$
$P = 4x$

$A = \left(\frac{P}{4}\right)^2 \Rightarrow A = \frac{P^2}{16}$

(b) The circumference of a circle as a function of its area A.

$C = 2\pi r$ —(i) and $A = \pi r^2$

$\frac{A}{\pi} = r^2 \Rightarrow r = \sqrt{\frac{A}{\pi}}$

putting in (i)

$C = 2\pi\sqrt{\frac{A}{\pi}} \Rightarrow 2\pi\frac{\sqrt{A}}{\sqrt{\pi}}$

$C = 2\sqrt{\pi}\sqrt{A} \Rightarrow C = 2\sqrt{\pi A}$

(c) The Surface area "S" of a Cube as a function of its volume "V":

$V = x^3$, $S \cdot A = x^2$ —— (i)

$(V)^{1/3} = (x^3)^{1/3}$

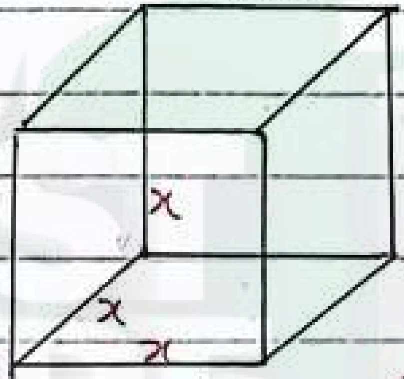
$x = V^{1/3}$ putting in (i)

$S = x^2$

$S = [V^{1/3}]^2$

$S = V^{2/3}$

Cube



$V = L \times B \times H$

$V = x \times x \times x$

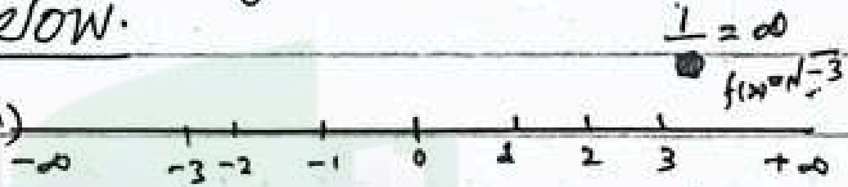
$V = x^3$

$S \cdot A = L \times B$

$= x \times x = x^2$

Q#4 Find the domain and Range of the function g defined below.

(i) $g(x) = 5 - x$ (Linear function)



Since $g(x)$ is defined for all real numbers so.

Domain: All real numbers OR $\mathbb{R}$ or $D_x = (-\infty, +\infty)$

Range: All real numbers or $\mathbb{R}$ or $R_x = (-\infty, +\infty)$

(ii) $g(x) = \sqrt{x+2}$

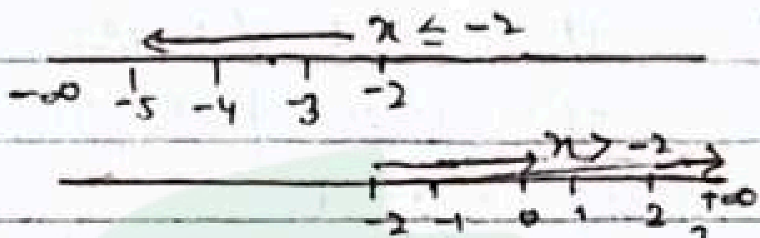
$g(x)$ is defined at any real numbers but it becomes complex when $x+2<0$ OR $x<-2$ so $x+2 \geq 0 \Rightarrow x \geq -2$

Domain: $[-2, +\infty)$

Range: $[0, +\infty)$

Range $= \sqrt{-2+2} \Rightarrow 0$

(iii) $g(x)=\begin{cases}6x+7, & x\le -2\\ 4x-3\cancel{x}, & x>-2\end{cases}$



Domain of $g(x)=6x+7$ is given as $x\le -2$, $(-\infty,-2]$

Domain of $g(x)=4-3x$ is given as $x>-2$ $(-2,+\infty)$

Domain:- $(-\infty,-2]\cup(-2,+\infty) \Rightarrow D_x=(-\infty,+\infty)$

Range of $g(x)=6x+7$ attains values $(-\infty,-5]$

Range of $g(x)=4-3x$ attains values $(-\infty,10)$

Range

$g(-2)=6(-2)+7$

$-12+7=-5$

$g(-2)=4-3(-2)$

$=10$

Range:- $(-\infty,-5]\cup(-\infty,10)$

$=(-\infty,10)$

(iv) $g(x) = |x-5|$

$-\infty$ —— $+\infty$

Since $g(x)$ is defined and real valued for all $x \in R$

Domain :- Set of all real numbers ~~or~~ $\mathbb{R}$ or $(-\infty, +\infty)$

Range :- Set of all real numbers non-negatives $\mathbb{R}^+$ OR $(0, +\infty)$

(v) $g(x) = \dfrac{x+2}{3-x}$

$\frac{1}{-1} = -1$

$g(x)$ is defined for all real numbers but

$(3-x) \neq 0 \Rightarrow x = 3$

Domain $= \mathbb{R} - 3$

Range $= \mathbb{R} - (-1)$

Q#5 Given $f(x) = x^3 - ax^2 + bx + 1$ if $f(2) = -3$ and $f(-1) = 0$ find the values of a and b.

Sol:- $f(x) = x^3 - ax^2 + bx + 1$ —(i)

put $x = 2$ in (i)

$f(2) = (2)^3 - a(2)^2 + b(2) + 1$

$= 8 - 4a + 2b + 1$

$-3 = -4a + 2b + 9$

$4a - 2b = 9 + 3$

$4a - 2b = 12$

$2(2a - b) = 12$

$2a - b = 6$ —(ii)

Now put $x = -1$ in (i)

$f(-1) = (-1)^3 - a(-1)^2 + b(-1) + 1$

$f(-1) = -1 - a - b + 1$

$0 = -a - b$ —(iii)

Subtracting Eq(ii) from (iii)

$$\begin{array}{r} -a - b = 0 \\ 2a - b = 6 \\ \underline{- \quad + \quad -} \\ -3a = -6 \end{array}$$

$\boxed{a = 2}$

put $a = 2$ in Eq (iii)

$-2 - b = 0$

$-2 = b$

$\boxed{b = -2}$

Q#6 A Stone falls a height of 60m on the ground, the height h after x second is approximately given by
$h(x) = 40 - 10x^2$ —— (i)

(i) What is the height of Stone when (a) $x = 1$ sec (b) $x = 1.5$ sec.

(a) put $x = 1$ in Eq(i)

$h(1) = 40 - 10(1)^2 = 30m$

(b) put $x = 1.5$ in (i)

$h(1.5) = 40 - 10(1.5)^2$
$= 40 - 22.5 \Rightarrow 17.5m$

(ii) When does the Stone Strike the ground?

put $h(x) = 0$ in (i)

$0 = 40 - 10x^2$

$10x^2 = 40$

$x^2 = \frac{40}{10} \Rightarrow 4$

$x^2 = 4$

$\sqrt{x^2} = \pm\sqrt{4} \Rightarrow x = \pm 2$

$\boxed{x = 2 \text{ sec}}$ $\quad x = -2$ (impossible)

Q#7 Consider the function $f(x) = 3x - 5$ —(i)

(i) Determine the domain and range of $f(x)$

It is a Linear function so its

Domain:- All real values or $\mathbb{R}$ or $(-\infty, +\infty)$

Range:- All real values or $\mathbb{R}$ or $(-\infty, +\infty)$

(ii) Is the function f one-to-one? Justify your answer

Yes it is a one-to-one function. For justisfication. We prove that $f(x_1) = f(x_2)$ for one-to-one.

$f(x) = 3x - 5$

put $x = x_1$ (i)

$f(x_1) = 3x_1 - 5$

put $x = x_2$

$f(x_2) = 3x_2 - 5$

By defination of one-to-one function $x_1 = x_2$ so

$3x_1 - 5 = 3x_2 - 5$

$3x_1 = 3x_2$

$x_1 = x_2$

Justified that the given function is one-to-one.

Q#7(iii) Is the function f onto if the Co-domain is all real numbers? Explain.

The function is onto if $f(x) = y$

$f(x) = 3x - 5$ —(i)

put $f(x) = y$ in (i)

$y = 3x - 5$

$y + 5 = 3x$

$\frac{y+5}{3} = x$

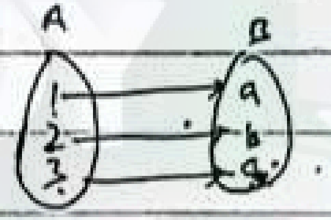
putting in (i)

$f(x) = 3\left[\frac{y+5}{3}\right] - 5$

$f(x) = y + 5 - 5$

$f(x) = y$

Hence proved that the given function is onto for all real numbers.



Q#8 Let $f : R \longrightarrow R$ be defined by $f(x) = \frac{2x-3}{x+1}$

(i) Find the domain and Range of $f(x)$

$f(x) = \frac{2x-3}{x+1}$

$\frac{2}{1} = 2$

put $x+1 \neq 0 \Rightarrow x \neq -1$

Domain:- All real numbers excluding -1 $R-(-1)$

Range:- All real numbers excluding 2

$R - \{2\}$ OR $(-\infty, 2) \cup (2, +\infty)$

(ii) Determine whether $f(x)$ is onto

The function is onto if $f(x) = y$

$f(x) = \frac{2x-3}{x+1}$ —— (i)

put $f(x) = y$ in (i)

Remaining (ii) Q#8

$$y = \frac{2x-3}{x+1}$$

$$y(x+1) = 2x-3$$

$$xy + y = 2x - 3$$

$$y + 3 = 2x - yx$$

$$y + 3 = x(2-y)$$

$$\frac{y+3}{2-y} = x$$

putting in (i)

$$f(x) = \frac{2x-3}{x+1} \quad \text{— (i)}$$

$$f(x) = \frac{2\left[\frac{y+3}{2-y}\right] - 3}{\left[\frac{y+3}{2-y}\right] + 1}$$

$$= \frac{2y+6}{2-y} - \frac{3}{1} \Big/ \frac{y+3}{2-y} + \frac{1}{1}$$

$$= \frac{2y+6-6+3y}{(2-y)} \Big/ \frac{y+3+2-y}{(2-y)}$$

$$= \frac{5y}{(2-y)} \times \frac{(2-y)}{5} \Rightarrow y$$

$f(x) = y$ Hence yes the function is onto.

iii) Prove that $f(x)$ is one-to-one.

For $f(x)$ is one-to-one

$f(x_1) = f(x_2)$ or

$x_1 = x_2$

$$f(x) = \frac{2x-3}{x+1} \quad \text{—— (i)}$$

put $x = x_1$ in (i)

$$f(x_1) = \frac{2x_1-3}{x_1+1}$$

put $x = x_2$ in (i)

$$f(x_2) = \frac{2x_2-3}{x_2+1}$$

For one-to-one function.

$$f(x_1) = f(x_2)$$

$$\frac{2x_1-3}{x_1+1} = \frac{2x_2-3}{x_2+1}$$

$$(2x_1-3)(x_2+1) = (2x_2-3)(x_1+1)$$

$$2x_1x_2 + 2x_1 - 3x_2 - 3 = 2x_1x_2 + 2x_2 - 3x_1 - 3$$

$$2x_1 - 3x_2 = 2x_2 - 3x_1$$

$$2x_1 + 3x_1 = 2x_2 + 3x_2$$

$$\cancel{5}x_1 = \cancel{5}x_2$$

$$x_1 = x_2$$

—— proved ——

Q#9 Consider the function $f: R^+ \longrightarrow R^+$ defined by $f(x) = e^{-x}$ Show that $f(x)$ is a bijective.

Sol:- bijective means we have to prove one-to-one and onto function:-

The function is one-to-one

The function is one-to-one if $f(x_1) = f(x_2)$ or $x_1 = x_2$

$$f(x) = e^{-x} \longrightarrow (i)$$

put $x = x_1$ in (i)

$$f(x_1) = e^{-x_1}$$

put $x = x_2$ in (i)

$$f(x_2) = e^{-x_2}$$

By the Condition

$$f(x_1) = f(x_2)$$

$$e^{-x_1} = e^{-x_2}$$

$$-x_1 = -x_2$$

$$x_1 = x_2 \quad \text{proved}$$

Now onto function

Condition $y = f(x)$

put $f(x) = y$ in (i)

$$y = e^{-x}$$

Taking ln on Both sides.

$\ln y = \ln e^{-x}$ —— (i)

$\ln y = -x \ln e$

$\ln y = -x \quad \because \ln e = 1$

put in (i) $\quad f(x) = e^{-x}$

$f(x) = e^{\ln y}$

$f(x) = y$ proved

Q#10 Let $g: \mathbb{R} \to \mathbb{R}$ is given by $g(x) = x^3 - 3x$. Find if $g(x)$ is injective or Surjective

For Surjective $g(x) = x^3 - 3x$

D_x = All real numbers $\Rightarrow (-\infty, +\infty)$

R_x = All real number $\Rightarrow (-\infty, +\infty)$

The function is Surjective. $f(x) = y$

Now Injective (one-to-one):

$g(x_1) = g(x_2) \Rightarrow x_1 = x_2$

$g(x_1) = x_1^3 - 3x_1$ and $g(x_2) = x_2^3 - 3x_2$

$g(x_1) = g(x_2)$ then

$x_1^3 - 3x_1 = x_2^3 - 3x_2$

$x_1^3 - x_2^3 = 3x_1 - 3x_2$

$(x_1 - x_2)(x_1^2 + x_1x_2 + x_2^2) - 3(x_1 - x_2) = 0$

$(x_1 - x_2)[(x_1^2 + x_1x_2 + x_2^2) - 3] = 0$

$x_1 - x_2 = 0 \Rightarrow x_1 = x_2$

$(x_1^2 + x_1x_2 + x_2^2 - 3) \neq 0$

So the given function is not injective (one-to-one)